Lesson 23

More properties of norm
6.1

Gram-Schmidt orthogonalization algorithm

6.2

Further properties of norm

Recall $\|v\| = \sqrt{\langle v\, v\rangle}$

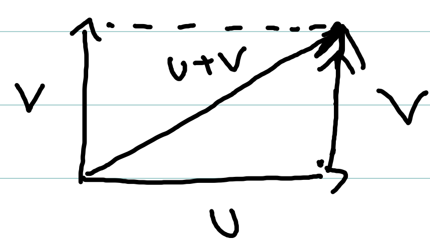
Th 1:

d) (Triangle inequality) $\|v+w\| \leq \|v\| + \|w\|$

$\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v+w\rangle + \langle w, v+w\rangle \quad =$

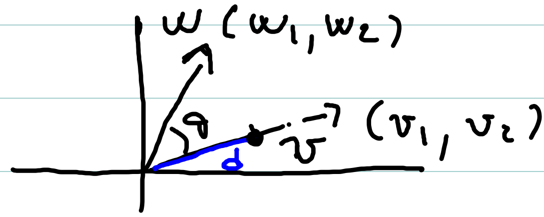
Def Let V be an inner product space. Then v and w are orthogonal (perpendicular) iff $\langle v, w \rangle = 0$

Pythagorean th: If u and v are orthogonal vectors $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

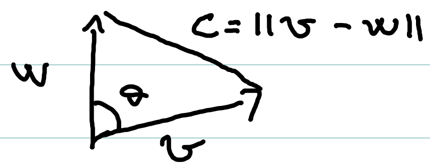


Hw problem

Note: in R^2 $\langle v, w \rangle = \|v\| \, \|w\| \cos\theta$



you can use the law of cosines



Def A set S of vectors is orthogonal if for any $v, w \in S$ if $v \neq w$ then $\langle v\ w \rangle = 0$

Def A set S of vectors is orthonormal if for every vector v in the list $\|v\| = 1$ and for every pair of vectors v, w with $v \neq w$ $\langle v, w \rangle = 0$

Def An orthonormal basis is a basis that is an orthonormal set

Th 2: An orthonormal set S is linearly independent. (more in general any orthogonal set of non zero vectors)

Th3: If $B = \{b_1, \dots b_n\}$ is an orthogonal set with all $b_i \neq 0$ and $v \in \text{Span}(B)$ then

$$v = \sum_{i=1}^{n} \frac{\langle v\, b_i \rangle}{\|b_i\|^2} b_i$$

Th4: if $B = b_1, b_2 \dots b_n$ is an orthonormal basis for V and $v \in V$ then $v = \sum_{i=1}^{n} \langle v\ b_i \rangle b_i$

Proof: by th3

Def: Let β be an orthonormal subset (possibly infinite) of V and $v \in V$ then the scalars $\langle v\ b \rangle$ for $b \in \beta$ are called Fourier coefficients of v relative to v

Idea: if V is infinite dimensional and has an orthonormal set $\{b_i \mid i \in \mathbb{N}\}$ and $v \in V$, try to write $v = \sum_{i=1}^{\infty} \langle v\ b_i \rangle b_i$.

Not a FINITE linear combination, to make sense of it need a notion of convergence:

$$\lim_{n \to +\infty} \sum_{i=1}^{n} \langle v\ b_i \rangle b_i$$

In a vector space V with a norm $\| \ \|$ we can define $\lim_{n \to +\infty} w_n = w$ iff

$$\forall \varepsilon > 0\ \exists m\ \forall n \geq m\ \| w_n - w \| < \varepsilon$$

Th Gram-Schmidt Every finite dimensional inner vector space has an orthonormal basis.

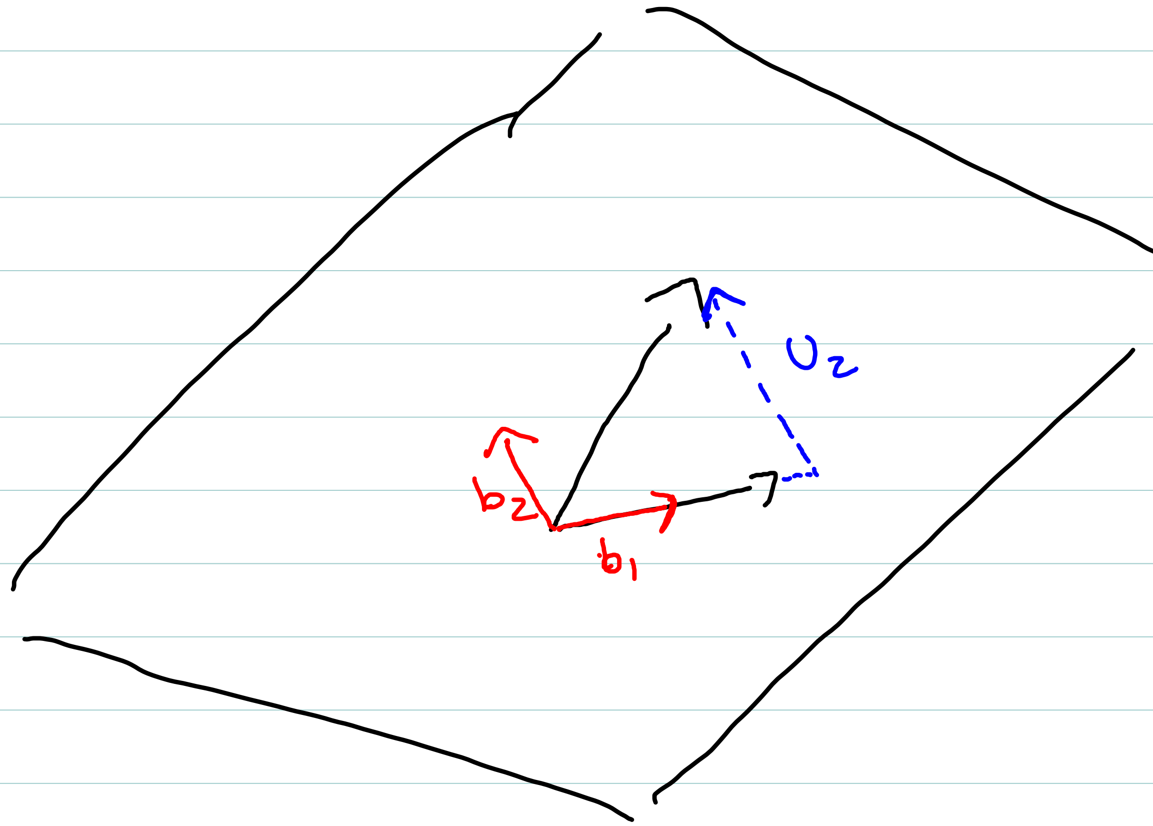
Proof:

Let $B_1 = w_1, \dots, w_n$ be a basis for V

We will define an orthonormal basis $B_2 = b_1, \dots, b_n$ by recursion as follows

Ex $W = \mathrm{span}((1\,1\,1)\ (2\,0\,1))$ plane in $\mathbb{R}^3$

$B = (1,1,1)\ (2,0,1)$ not orthonormal



$$\langle \underset{v}{(1\ 1\ 1)} \quad \underset{w}{(2\ 0\ 1)} \rangle = 3$$

$$3 = \sqrt{3} \cdot \sqrt{5} \cos\theta$$

$$\sqrt{\frac{3}{5}} = \cos\theta \qquad \theta \approx 0.7 \text{ red}$$

Example consider $P_2(R)$ as a vector space over R

with $\langle p, q\rangle = \int_{-1}^{1} p\, q\, dx$

Check $\langle\ \rangle$ is an inner product and find orthonormal basis for $P_2(R)$

1) $\langle p+q, f\rangle = \int_{-1}^{1}(p+q) f\, dx = \int_{-1}^{1} p f\, dx + \int_{-1}^{1} q f\, dx = \langle p\ f\rangle + \langle q\ f\rangle$

2) $\langle cp, f\rangle = \int_{-1}^{1} c p f\, dx = c\int_{-1}^{1} p f\, dx = c\langle p\ f\rangle$

3) $\langle p\ q\rangle = \int_{-1}^{1} pq = \int_{-1}^{1} qp = \langle q\ p\rangle$

4) $\langle p\ p\rangle = \int_{-1}^{1} p^2\, dx > 0$ if $p \neq 0$

if $p \neq 0$ $p(x_0) \neq 0$ for some x_0 so

$p^2(x_0) > 0$ in some interval $(x_0 - \varepsilon\ \ x_0 + \varepsilon)$

Therefore $\int_{-1}^{1} p^2\, dx > 0$

Find an orthonormal basis for $P_2(R)$

Start with $\{1, x, x^2\}$ and apply Gram Schmidt:

$$\|1\|^2 = \int_{-1}^{1} 1\cdot 1\, dx = x\Big|_{-1}^{1} = 2$$

$$b_1 = \frac{1}{\sqrt{2}}\cdot 1$$

$$v_2 = x - \langle x\, b_1\rangle b_1$$

$$\langle x, \frac{1}{\sqrt{2}}\rangle = \int_{-1}^{1} x\cdot\frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}\frac{x^2}{2}\Big|_{-1}^{1} = 0$$

$$\|x\|^2 = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$b_2 = \sqrt{\frac{3}{2}}\, x$$

$$v_3 = x^2 - \langle x^2\, b_1\rangle b_1 - \langle x^2\, b_2\rangle b_2$$

$$\int_{-1}^{1} x^2 \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{\sqrt{2}\,3}$$

$$\int_{-1}^{1} x^2\cdot\sqrt{\frac{3}{2}}\, x\, dx = 0$$

$$v_3 = x^2 - \frac{2}{\sqrt{2}\,3}\cdot\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$$

$$\|v_3\|^2 = \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} =$$

$$= \frac{x^5}{5}\Big|_{-1}^{1} - \frac{2}{9}x^3\Big|_{-1}^{1} + \frac{1}{9}x\Big|_{-1}^{1} = \frac{2}{5} - \frac{4}{9} +$$

$$= \frac{8}{45} \qquad v_3 = 3\sqrt{\frac{5}{8}}\left(x^2-\frac{1}{3}\right)$$

Orthonormal basis is $\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, 3\sqrt{\frac{5}{8}}\left(x^2-\frac{1}{3}\right)$