Lesson 22

Inner product spaces

read 6.1

Complex numbers

$z = x + iy \qquad x = \mathrm{Re}\, z \qquad y = \mathrm{Im}\, z$

Usual arithmetic and $i^2 = -1$

Ex $(2+3i) - (1-i) = 1 + 4i$

$(2+3i)(1-i) = 2 - 2i + 3i - 3i^2 = 5 + i$

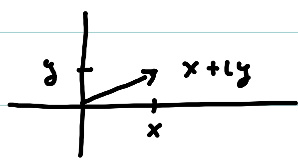
$z = x+iy \qquad \bar{z} = x - iy$ (conjugate of z)

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2+y^2}$ Note $|z| \in R \qquad |z| \geq 0$

norm of z or modulus of z



if $x \in R \qquad \bar{x} = x$

Inner product spaces

Def Let V be a vector space over F. An inner product on V is a function $f\ V\times V \to F$ s.t the following hold.

a) $f(x+z, y) = f(x,y) + f(z,y)$

b) $f(cx, y) = c\, f(x,y)$

c) $f(x\ y) = \overline{f(y,x)}$ (if $F=\mathbb{R}$ this becomes $f(x\ y) = f(y,x)$)

d) $f(x,x) > 0$ if $x \neq 0$ (note that $f(x\ y) = \overline{f(y\ x)}$ implies $f(x\ x) \in \mathbb{R}$ even when $F = \mathbb{C}$)

Th The following are true in an inner product space V

a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$:

b) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$:

c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$:

d) $\langle x, x \rangle = 0$ iff $x = 0$:

e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$ then $y = z$

Def Let V be an inner product space. Then $\|v\| = \sqrt{\langle v \; v \rangle}$.

Th Let V be an inner product space over F. Then for all $v, w \in V$ and $c \in F$ we have

a) $\|cv\| = |c| \cdot \|v\|$ ($|c|$ = absolute value if $c \in \mathbb{R}$, modulus if $c \in \mathbb{C}$
$|x+iy| = \sqrt{x^2+y^2}$)

b) $\|v\| \geq 0$

c) $\|v\| = 0 \iff v = 0$

d) Cauchy Schwartz : $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$

Use Cauchy Schwartz inequality to prove

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$$

for all positive integers n and real numbers x_i

Further properties of norm

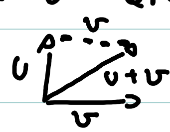
Recall $\|v\| = \sqrt{\langle v\, v \rangle}$

d) (Triangle inequality) $\|v+w\| \leq \|v\| + \|w\|$

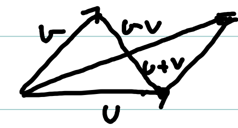
.

Def Let V be an inner product space. Then v and w are orthogonal (perpendicular) iff $\langle v, w \rangle = 0$

Pythagorean th: If u and v are orthogonal vectors $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.



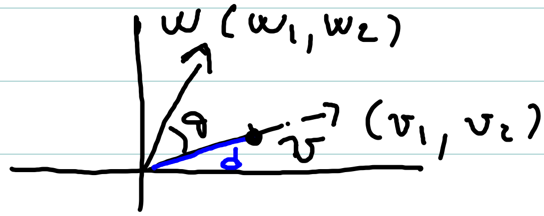
Th (Parallelogram equality)



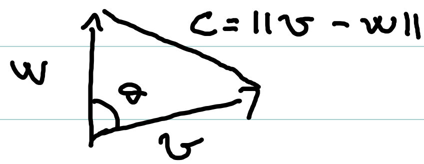
(The sum of the squares of the lengths of the diagonals of a parallelogram are equal to the sum of the lengths of the sides): $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

Angles:

Note: in R^2 $\langle v, w\rangle = \|v\|\,\|w\| \cos\theta$



you can use the law of cosines



$$\|w\|^2 + \|v\|^2 = \|v-w\|^2 + 2\|v\|\,\|w\| \cos\theta$$

$$w_1^2 + w_2^2 + v_1^2 + w_2^2 = (v_1 - w_1)^2 + (v_2 - w_2)^2 + 2\|v\|\|w\|\cos\theta$$

$$2v_1 w_1 + 2v_2 w_2 = 2\|v\|\,\|w\| \cos\theta$$

$$\langle v \quad w\rangle = \|v\|\,\|w\| \cos\theta$$

Note: it is possible to define a norm on a vector space over F ($\mathbb{R}$, $\mathbb{C}$) as a function $\| \ \|$ $V \to \mathbb{R}^+$ with the properties

1) $\|v\| = 0 \iff v = 0$
2) $\|cv\| = |c| \, \|v\|$
3) $\|v+w\| \leq \|v\| + \|w\|$

There are vector spaces with a norm $\|v\|$ that cannot be defined from a $\langle \ , \ \rangle$

Ex $\| \ \|$ $\mathbb{R}^2$

$\|(x\ y)\| = \max\{|x|, |y|\}$ is a norm but it does not satisfy the parallelogram law, so it does not come from an inner product.