Lesson 21

More about Generalized eigenvectors.

Th 1: Assume $T \in \mathcal{L}(V)$

$\dim V = n$, then

$V = N(T^n) \oplus R(T^n)$

Th 2: Assume $T \in \mathcal{L}(V)$, $\dim V = n$, $F = \mathbb{C}$, then the generalized eigenvectors of T span V

Proof by induction on n

Th 3: Let $T \in \mathcal{L}(V)$, $\dim V = n$, $F = \mathbb{C}$ and let $\lambda_1, \lambda_2, \dots \lambda_m$ be the distinct eigenvalues for T

Then :

(1) $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_m}$

(2) K_{λ_i} is invariant for T for all $i = 1, \dots m$

(3) There is a basis B of V made of generalized eigenvectors of T such that T_B^B is a block matrix

$$T_B^B = \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_m \end{bmatrix}$$

Each matrix A_i is a $n_i \times n_i$ matrix, where n_i is the dimension of K_{λ_i}

$$A_i = \begin{bmatrix} \lambda_i & * & & * \\ 0 & \ddots & & * \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

In particular T_B^B is upper triangular

Let's choose bases B_i for K_{λ_i} in the following way:

remember that

$$E_{\lambda_i} = N(T-\lambda_i I) \subset N(T-\lambda_i I)^2 \subset N(T-\lambda_i I)^3 \cdots \subset N(T-\lambda_i I)^{h_i} = N(T-\lambda_i I)^{h_i+1} = \cdots = \underbrace{N(T-\lambda_i I)^n}_{K_{\lambda_i}}$$

$v_1 \cdots v_k$ basis for $N(T-\lambda_i I)$

$v_1 \cdots v_k \; v_{k+1} \cdots v_{k+j_1}$ basis for $N(T-\lambda_i I)^2$

$v_1 \cdots v_k \; v_{k+1} \cdots v_{k+j_1} \; v_{k+j_1+1} \cdots v_{k+j_1+j_2}$ is a basis for $N(T-\lambda_i I)^3$

$$\underbrace{v_1 \cdots v_k}\; \underbrace{v_{k+1} \cdots v_{k+j_1}}\; \underbrace{v_{k+j_1+1} \cdots v_{k+j_1+j_2}} + \cdots\cdots \; v_{n_i}$$

is a basis for K_{λ_i}

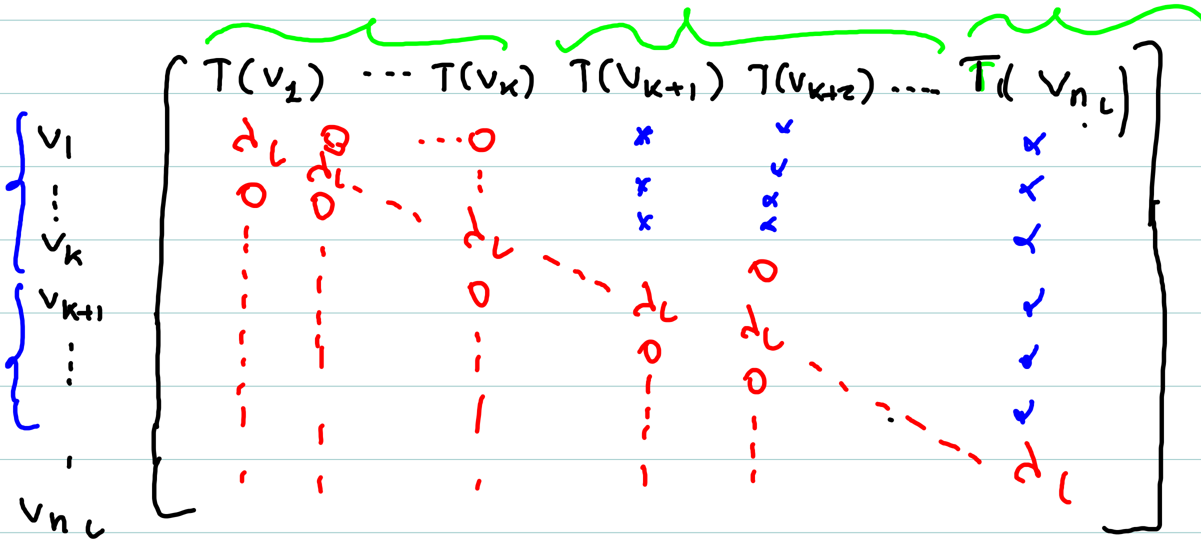
$B = \bigcup B_i$

What does T_B^B look like?

Recall each K_{λ_i} is T invariant

$$T_B^B = \begin{array}{c c c}
 & & \begin{array}{cccc} \overbrace{T(b_1) \cdots}^{B_1} & \overbrace{T(b_2)}^{B_2} & \cdots\cdots & T(b_n) \end{array} \\
\begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_m \end{array} & \begin{array}{c} b_1 \\ \vdots \\ \vdots \\ b_n \end{array} & \begin{bmatrix} \boxed{A_1} & 0 & 0 \\ 0 & \boxed{A_2} & 0 \\ 0 & 0 & \boxed{A_m} \end{bmatrix} \\
 & & \begin{array}{cccc} \underbrace{\qquad}_{B_1} & \underbrace{\qquad}_{B_2} & \cdots & \underbrace{\qquad}_{B_m} \end{array}
\end{array}$$

What does A_L look like?



$$T(v_1) = \lambda_L v_1 \cdots \; T(v_k) = \lambda_L v_k$$

$$T(v_{k+1}) = \underbrace{(T-\lambda_L I)\, v_{k+1}}_{\in N(T-\lambda_L I)} + \lambda_L v_{k+1}$$

$$T(v_{k+j_1+1}) = \underbrace{(T-\lambda_L I)\, v_{k+j_1+1}}_{\in N(T-\lambda_L I)^2} + \lambda_L I\, v_{k+j_1+1}$$

$$T(v_{n_L}) = \underbrace{(T-\lambda_L I)\, v_{n_L}}_{\in N(T-\lambda_L I)^2} + \lambda_L I\, v_{n_L}$$

n_i is the dim of K_{λ_i}

n_i is the algebraic multeplicity of λ_i

$n_i \geq h_i$ so $K_{\lambda_i} = N(T - \lambda I)^{h_i}$

Th 4: : Let $T \in \mathcal{L}(V)$ $\dim V = n$

λ an eigenvalue of T with algebraic multiplicity m. Then $\dim(K_\lambda) = m$

and $K_\lambda = N(T - \lambda I)^m$

Proof Consider the proof of the previous th where $\dim(K_{\lambda_i}) = n_i$ and all the vectors in the chosen basis of K_{λ_i} are in $N(T - \lambda_i)^{h_i}$. $n_i \geq h_i$ so $K_{\lambda_i} = N(T - \lambda_i)^{n_i}$. T_B^B in the proof of the previous th. has n_i λ_i on the diagonal and it has characteristic polynomial
$(x - \lambda_1)^{n_1} (x - \lambda_2)^{n_2} \cdots (x - \lambda_m)^{n_m}$

so $n_i =$ algebraic multiplicity of λ_i

Jordan canonical form

For a better chosen B

T_B^B has the form

$$\begin{bmatrix} d_1 & * & & & & & & & \\ & \ddots & * & & & & & 0 & \\ & & d_1 & * & & & & & \\ & & & d_2 & * & & & & \\ & & & & \ddots & * & & & \\ & 0 & & & & \ddots & * & & \\ & & & & & & \ddots & * & \\ & & & & & & & d_m & \end{bmatrix}$$

$*$ can be 0 or 1