Lesson 20

Linear operators (matrices) over $\mathbb{C}$ - Generalized eigenspaces

see ch 7

Recap: given an $n \times n$ matrix A with entries in F (so here $V = F^n$) or an operator $T: V \to V$ with $\dim(V) = n$

If $\lambda_1, \lambda_2, \dots \lambda_m$ are the distinct eigenvalues of A/T, A/T is diagonalizable iff

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m}$$

Problem 1

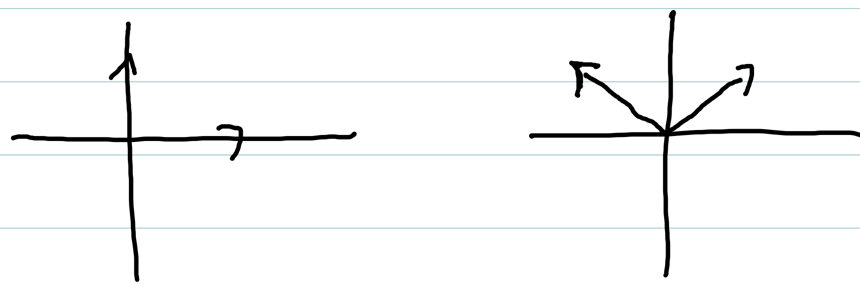
$F = \mathbb{R}$

$A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ has no real eigenvalues

Geometric interpretation

$$A = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$$

rotation



No fixed lines = no eigenvalues

$$p(x) = (1-x)^2 + 1$$

In $\mathbb{C}$ $\quad p(x) = (x-(1+i))(x-(1-i))$

Th (Fundamental th of Algebra)

every polynomial $p(x)$ splits over $\mathbb{C}$

Corollery 1 : Every $T \in \mathcal{L}(V)$

where $\dim V = n$

vector spece over $\mathbb{C}$ hes at

least one eigenvelue

Proof Let $p(x)$ be the clarecteristic polynomiel of T then

$$p(x) = c\,(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

so T hes at least one eigenvelue

(it could be only one if $\lambda_1 = \lambda_2 \cdots = \lambda_n$)

Example of $T \in \mathcal{L}(V)$, V a complex vector space. T has no eigenvalues.
V is not finite dimensional.

$$T \quad \mathbb{C}^{\infty} \longrightarrow \mathbb{C}^{\infty}$$

$$T(x_1\ x_2 \cdots x_n \cdots) = (0\ x_1\ x_2 \cdots)$$

Problem 2

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore it is possible that

$$E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m} \lneq V$$

If we want to write V as sum of invariant subspaces of T, we need more vectors.

Def : Let $T \in \mathcal{L}(V)$ and λ_i be an eigenvalue for T, w is a generalized eigenvector for λ_i if $w \neq 0$ and $w \in N(T-\lambda_i I)^k$ for some k in $\mathbb{N}$

Note : if $w \in N(T-\lambda_i I)^k$ and k is the smallest positive integer such that this is true, then $(T-\lambda_i I)^{k-1} w$ is an eigenvector for λ_i

Th 1:

1) $N(T-\lambda I)^k \subseteq N(T-\lambda I)^{k+1}$ for all $k \geq 1$

2) if $N(T-\lambda I)^k = N(T-\lambda I)^{k+1}$

then $N(T-\lambda I)^k = N(T-\lambda I)^{k+m}$

for all $m \geq 1$

Note: If $\dim V = n$ and $\dim E_\lambda = p$

$$E_\lambda = \underset{\dim p}{N(T-\lambda I)} \subset \underset{\substack{\text{dim at} \\ \text{least } p+1}}{N(T-\lambda I)^2} \subset \cdots \subset \underset{\substack{\text{dim at least} \\ p+k-1}}{N(T-\lambda I)^k}$$

So we may have proper inclusion but eventually we need to stop because V has finite dimension

Th2: If $T \in \mathcal{L}(V)$, $\dim V = n$, λ is an eigenvalue for V, then any generalized eigenvector for λ is in $N(T-\lambda I)^n$

Proof: by the discussion above

Note: the discussion above did not use the fact λ is an eigenvalue for T, so it is true that if $T \in \mathcal{L}(V)$ $\dim V = n$

$$N(T^n) = N(T^{n+k}) \quad \text{for all } k \geq 0$$

Def: If $T \in \mathcal{L}(V)$, $\dim V = n$, λ is an eigenvalue for V, then we call $N(T - \lambda I)^n$ the generalized eigenspace of λ and denote it K_λ

Th 3 Assume $T \in \mathcal{L}(V)$, $\dim(V) = n$ and λ is an eigenvalue of V

Then K_λ is T invariant

Th4: Suppose $T \in \mathcal{L}(V)$, $\dim V = n$, $\lambda_1, \lambda_2, \dots \lambda_m$ are distinct eigenvalues for T and $w_1, w_2, \dots w_m$ are corresponding generalized eigenvectors. Then $w_1, w_2, \dots w_m$ are linearly independent.

Th5: Suppose $T \in \mathcal{L}(V)$, $\dim V = n$, $\lambda_1, \lambda_2, \cdots \lambda_m$ are distinct eigenvalues for T then The sum

$$K_{\lambda_1} + K_{\lambda_2} + \cdots + K_{\lambda_m} \text{ is direct}$$

Proof: suppose $0 = v_1 + \cdots + v_m$ with $v_i \in K_{\lambda_i}$. If any of these vectors were non zero they would be linearly independent and add up to 0, impossible