Lesson 19

5.2 the charecteristic polynomiel

Def If A is an $n \times n$ matrix
$\det(A - xI) = p(x)$ is called
the characteristic polynomial of A

Note the eigenvalues of A
are the solutions of $p(x) = 0$

Def If $T \in \mathcal{L}(V)$, V is finite
dimensional and B is a basis
for V then the characteristic
polynomial of T is the characteristic
polynomial of T^B_B

Note: What happens if C
is another basis for V?

Th 1 If A is a $n \times n$ matrix with entries in $P_1(\mathbb{R})$ $\det A$ is a polynomial in $P_n(\mathbb{R})$

Proof: by induction on n

Base case: if $n=1$ $A = (a\lambda + b)$
$\det(A) = a\lambda + b$

Induction step:

Th 2: If $A = (a_{ij})$ is $n \times n$ $\det(A - XI)$ is a polynomial in X of degree n
$p(X) = (a_{11} - X)(a_{22} - X) \cdots (a_{nn} - X) + g(X)$
where $g(X) \in P_{n-2}(R)$

proof by induction on n

Base case: if $n = 2$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det(A - XI) = (a_{11} - X)(a_{22} - X) - a_{12} a_{21}$

Induction step: assume the theorem is true for n and A is $(n+1) \times (n+1)$

$$B = A - \lambda I = \begin{pmatrix} a_{11} - X & a_{12} & \cdots & a_{1j} & \cdots & a_{1\,n+1} \\ a_{21} & a_{22} - X & & a_{2j} & & a_{2\,n+1} \\ \cdots & & & & & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} - X & \cdots & a_{j\,n+1} \\ & & & & & \\ a_{n+1\,1} & \cdots & & a_{n+1\,j} & & a_{n+1\,n+1} - X \end{pmatrix}$$

Definition: a polynomial $p(x)$ in $P(F)$ splits over F if we can write $p(x) = c\,(x-a_1)(x-a_2)\cdots(x-a_n)$
$a_1, a_2, \dots a_n$ are called the roots of p, they are not necessarily all distinct.

Th 3: If $T \in \mathcal{L}(V)$, where $\dim V < +\infty$ is diagonalizable, then $p(x)$, the characteristic polynomial of T splits

Def : Let λ be an eigenvalue for an operator or a matrix then $\dim(E_\lambda)$ is called the geometrix multeplicity of λ. If $p(x)$ is the characteristic polynomiel of the operator / matrix the largest positive k s.t $(x-\lambda)^k$ is a factor of $p(x)$ is called the algebraic multeplicity of λ

Th 4: Let $T \in \mathcal{L}(V)$, $\dim V < +\infty$ λ an eigenvalue for T. Then the geometric multeplicity of λ is less then or equal to the algebraic multeplicity of λ

Th 5 : Let $T \in \mathcal{L}(V)$ $\dim V = n$ and $F = \mathbb{C}$

Let $\lambda_1 \ldots \lambda_k$ be the distinct eigenvalues of T. Then T is diagonalizable iff for all i the geometric multeplicity of λ_i is equal to the algebraic multeplicity.

Proof: assume T is diagonalizable and $\dim(E_{\lambda_i}) = p_i$

$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ therefore

$n = p_1 + p_2 + \cdots + p_k$

so the charecteristic polynomial must be $\pm (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2}\ldots(x-\lambda_k)^{p_k}$

assume that for all i

p_i = algebraic multeplicity of λ_i

then $p(x)$ splits in the product of linear factors and we have p_1 factors equal to $(x-\lambda_1)$, p_2 factors equal to $(x-\lambda_2)\ldots$ therefore

$p_1 + p_2 + \cdots + p_k = n$ so

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots + E_{\lambda_k}.$$

Example $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$$T(p) = p'(x)(x+1)$$

Is T diagonalizable?