Lesson 17

Invariant subspaces

Eigenvalues and eigenvectors.

read 5.1

Def: A linear transformation $T\ V \to V$ is called a (linear) operator

Def: A linear transformation $T\ V \to F$ is called a (linear) functional

Def: A linear operator T on a finite dimensional vector space V is called diagonalizable if V has an ordered basis B s.t T_B^B is a diagonal matrix

Def A $n \times n$ matrix is called diagonalizable if $T : R^n \to R^n$

$$Tx = Ax$$

is diagonalizable.

$A = T_E^E$ where E is the canonical basis

$$T_B^B = I_E^B \; T_E^E \; I_B^E$$

$$D = P^{-1} A P$$

$$P D P^{-1} = A$$

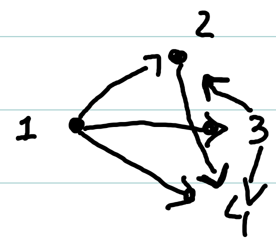
very useful matrix decomposition

Goal (ch 5 and 6)

A $n \times n$ matrix with complex entries is diagonalizable iff
$A\,\overline{A^t} = \overline{A^t} \cdot A$ (A is normal)

An $n \times n$ matrix with real entries is diagonalizable iff $A = A^t$
(A is symmetric)

1 Million doller eigenvector.



	1	2	3	4
1				
2				
3				
4				

Def: Suppose $T \in \mathcal{L}(V)$ and $U \leq V$. U is invariant under T if $T(U) \leq U$ i.e $u \in U \Rightarrow T(u) \in U$

Ex 1: Given $T\ V \to V$, $N(T)$, $R(T)$, V, $\{0\}$ are all invariant under T,

Ex2: for a fixed $v \in V$

$$W = \text{span}\{T^n x \mid n \geq 0\} = \text{span}\{x, Tx, T^2x \dots\}$$

is invariant under T

Note: if $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $T \in \mathcal{L}(V)$ and $T(U_i) \leq U_i$ then in order to understand T we just need to understand $T_{U_j} : U_j \to U_j$ for $j = 1 \cdots n$

The easiest situation is when each U_i has dimension 1

Ex3: Consider $T \; V \to V$

what are the dim 1 invariant subspaces ?

$U = \text{span}(v)$ $T(v) \in \text{Span}(v) \iff T(v) = \lambda v$

for some $\lambda \in F$

Def Given $T \in \mathcal{L}(V)$ a non zero vector s.t

$T(v) = \lambda v$ is called an <u>eigenvector</u> of T

λ is the associated <u>eigenvalue</u>

Ex 4 : given $T \in \mathcal{L}(P(\mathbb{R}))$ $T(p) = p'$

What are the eigenvalues and eigenvectors of T ?

If $T(p) = p' = \lambda p$ then p and p' are polynomials

of the same degree. This happens only if $p = c$ and $\lambda = 0$

Note if $T \in \mathcal{L}(V)$ and $\beta = \{v_1, v_2, \dots v_n\}$

is a basis for V then T_β^β is diagonal iff

each v_i is an eigenvector for T

in this case $V = \text{span}(v_1) \oplus \text{span}(v_2) \oplus \dots \oplus \text{span}(v_n)$

Def: If A is an $n \times n$ matrix eigenvectors eigenvalues of A are eigenvectors eigenvalues for L_A $R^n \to R^n$ $L_A(v) = Av$

i.e λ is an eigenvalue for A if there is a non zero vector v in R^n s.t

$Av = \lambda v$. Such a v is called an eigenvector for A

Th 1: Given $T \in \mathcal{L}(V)$, where V is a finite dimensional vector space, the following are equivalent:

1) λ is an eigenvalue for T
2) $T - \lambda I$ is not injective :
3) $T - \lambda I$ is not surjective
4) $T - \lambda I$ is not invertible

Proof 1) $\Leftrightarrow$ 2) λ is an eigenvalue for T iff $T(v) = \lambda v$ for some $v \neq 0$ $v \in V$ that is iff $(T - \lambda I)v = 0$ iff $N(T - \lambda I) \neq \{0\}$ iff $T - \lambda I$ is not injective.

We already know 2) 3) 4) are equivalent (using $\dim V = \dim N(T-\lambda I) + \dim R(T - \lambda I)$)

Note: eigenvectors for λ are the non zero vectors in $N(T - \lambda I)$

Thz If A is an $n \times n$ matrix
λ is an eigenvalue of A iff $\det(A-\lambda I)=0$

Proof assume $v \neq 0$ and $Av = \lambda I$
then $(A-\lambda I)v = 0$ so $A-\lambda I$ is
not invertible (308 stuff: columns
of $A-\lambda I$ are dependent) so
$\det(A-\lambda I) = 0$

vice-verse if $\det(A-\lambda I)=0$
then $A-\lambda I$ is singuler and
therefore the system $(A-\lambda I)x=0$ has
infinitely many solutions so
there must be some $v\neq 0$ s t
$(A-\lambda I)v=0$ so $A=\lambda v$

Def $\operatorname{Det}(A-xI)=p(x)$
is called the characteristic polynomial
of A

Th 3 If A is a $n\times n$ metrix
with entries in $P_1(\mathbb{R})$ $\det A$ is a
polynomial in $P_n(\mathbb{R})$

Proof: by induction on n

Base case: if $n=1$ $A=(a\lambda+b)$
$\det(A)=a\lambda+b$

Induction step: assume the theorem is true for n and A is a $(n+1)\times(n+1)$ matrix

$\det A = \sum_{j=1}^{n} (-1)^{1+j} \underbrace{a_{1j}}_{\in P_1(R)} \underbrace{\det(A_{1j})}_{\in P_n(R)}$ is sum

of polynomials in $P_{n+1}(R)$ so it is a polynomial in $P_{n+1}(R)$

Th 4 If A is $n\times n$ $\det(A-\lambda I)$ is a polynomial in λ of degree n

$p(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + g(\lambda)$

where $g(\lambda) \in P_{n-2}(R)$

proof by induction on n λ

Base case: if $n=2$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$

Induction step: assume the theorem is true for n and A is $(n+1)\times(n+1)$

$$B = A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n+1} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ a_{n+1 1} & \cdots & & a_{n+1\, n+1}-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (a_{11}-\lambda)\det(B_{11}) +$$

$$+ \sum_{j=2}^{n+1} (-1)^{1+j} a_{1j} \det(B_{1j})$$

$B_{11} = (A_{11} - \lambda I)$ by induction assumption $\det(B_{11}) =$

$$(a_{22}-\lambda)(a_{33}-\lambda)\cdots(a_{nn}-\lambda) + \underbrace{h(\lambda)}_{\in P_{n-2}(R)}$$

so $(a_{11}-\lambda)\det(B_{11}) = (a_{11}-\lambda)\cdots(a_{nn}-\lambda) + \underbrace{(a_{11}-\lambda)h(\lambda)}_{\in P_{n-2}(R)}$

What about $(-1)^{1+j} a_{1j} \det(B_{1j})$ for $j > 1$?

$$B_{1j} = C = \begin{pmatrix} a_{22} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n+1\,1} & a_{n+1\,2} & & a_{n+1\,n+1}-\lambda \end{pmatrix} \in P_{n-2}(R)$$

if we expend along j row we get

$$\sum_{k \neq j} (-1)^{1+j} a_{1j} \, (-1) \, (-1)^{j+k} a_{kj} \underbrace{\det(C_{jk})}_{\in P_{n-1}(R)}$$

Example:

$T \quad P_1(R) \longrightarrow P_1(R)$

$T(a+bx) = a+2b+3bx$

$B = \{1, x\}$

$$T_B^B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda)$$

eigenvalues $\lambda = 1$, $\lambda = 3$

eigenvectors for $\lambda = 1$: solve

$$\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x = 1 \\ y = 0 \end{matrix}$$

$(1, 0)$ is eigenvector for T_B^B

$p(x) = 1 + 0x$ is eigenvector for T

eigenvectors for $\lambda = 3$: solve

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = 1$
$y = 1$

$v = (1, 1)$ is eigenvector for T_B^B

$p(x) = 4x$ is eigenvector for T

Quiz consider T.

B_1' $\quad T_{B_1}^{B_1}$ has same eigenvalues yes

$T_{B_1}^{B_1}$ has same eigenvectors no

$$T_{B_1}^{B_1} = I_B^{B_1} T_B^B I_{B_1}^B$$

$$Tv = \lambda v$$

$$T_{B_1}^{B_1} [v]_{B_1} = [\lambda v]_{B_1} = \lambda [v]_{B_1}$$

$$T_B^B [v]_B = [\lambda v]_B \quad \lambda [v]_B$$

$$Tv = \lambda v$$

$B_1 = 1,\ 1+x$ Skip

$$T_{B_1}^{B_1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

What is

$$T^{10}(x)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3^{10} \end{bmatrix} \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{[x]_{B_1}} = \underbrace{\begin{pmatrix} -1 \\ 3^{10} \end{pmatrix}}_{[T(x)]_{B_1}}$$

$$T^{10}(x) = -1 + 3^{10}(1+x) = 3^{10} - 1 + 3^{10}x$$

$$T_B^B = I_{B_1}^B \ T_{B_1}^{B_1} \ I_B^{B_2}.$$

$$T_B^B = I_{B_1}^B \ T_{B_1}^{B_1} \left(I_{B_1}^B\right)^{-1}$$

$$\left(T_B^B\right)^{10} = I_{B_1}^B \left(T_{B_1}^{B_1}\right)^{10} \left(I_{B_1}^B\right)^{-1}$$

Example: $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$

$T(a+bx) = 2a+3bx$

$T(1) = 2 \cdot 1$

$T(x) = 3 \cdot x$

T has eigenvalues 2, 3

1 is an eigenvector of T with eigenvalue 2

$(T-2I)(a+bx) = 2a+3bx-2a-2bx = bx$

$N(T-2I) = \{c \mid c \in \mathbb{R}\}$ constant polynomials

x is an eigenvector of T with eigenvalue 3

$(T-3I)(a+bx) = 2a+3bx-3a-3bx = -a$

$N(T-3I) = \{bx \mid b \in \mathbb{R}\}$

$T \quad V \to V \qquad \mathcal{B}$ basis for V

λ is an eigenvalue for T with eigenvector v $\iff$ λ is an eigenvalue for $T^{\mathcal{B}}_{\mathcal{B}}$ with eigenvector $[v]_{\mathcal{B}}$

$T v = \lambda v \qquad T^{B}_{\mathcal{B}} [v]_{\mathcal{B}} = [T v]_{\mathcal{B}} = [\lambda v]_{\mathcal{B}} = \lambda [v_{\mathcal{B}}]$

vice versa

if $T^{B}_{\mathcal{B}} w = \lambda w$ $\qquad$ and $w = (x_1 \cdots x_n)$

Then $T V = \lambda V.$ $\qquad$ $V = x_1 b_1 + \cdots x_n b_n$

$w = [V]_{\mathcal{B}}$

$$T^{B_2}_{\mathcal{B}_2} = I^{B_2}_{B_1} \; T^{B_1}_{B_1} \; I^{B_1}_{B_2} = P^{-1} \; T^{B_1}_{B_1} \; P$$

so all $T^{B}_{\mathcal{B}}$ have the same char poly: $\qquad P^{-1} A P - \lambda I$:

$= P^{-1} A P - P^{-1} \lambda I P = P^{-1}(A - \lambda I) P$

same eigenvalues λ $\quad$ what abot eigenvectors and E_λ ? $\quad$ They all have same dim