Lesson 15

Determinants

ch 4

Determinants

Th1: Any such D has the following properties:

a) $D(r_1 \cdots r_i \cdots r_j \cdots r_n) = -D(r_1 \cdots r_j \cdots r_i \cdots r_n)$

b) $D(r_1 \cdots k r_i \cdots r_n) = k D(r_1 \cdots r_i \cdots r_n)$

c) $D(r_1 \cdots r_i + k r_j \cdots r_j \cdots r_n) = D(r_1 \cdots r_i \cdots r_j \cdots r_n)$

d) If M is singular i.e $r_1 \cdots r_n$ are dependent then $D(M) = 0$

e) If M is non singular then $D(M) \neq 0$

Th 2 (Uniqueness):

Th 4: you can expand along any row

Th 3 (Existence) define D_n $M_{n\times n} \to R$

$$D_n(M) = \sum_{i=1}^{n} (-1)^{1+j} A_{1j} \det(A_{1j})$$

$$D_1(a) = a$$

The determinent function from 308 (No proof. See th 4.3 in the text)

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \\ & & & & \end{pmatrix}$$

Expanding along first row

Notes on matrix multiplication

$$\underset{m\times n}{A}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 c_1 + x_2 c_2 + \cdots x_n c_n$$

$$(y_1\, y_2 \cdots y_m)\, A = y_1 r_1 + y_2 r_2 + \cdots + y_m r_m$$

$$A\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = c_1 \qquad (1 \cdots 0)\, A = r_1$$

$$\underset{m\times n}{A} \cdot \underset{n\times k}{B} \quad \begin{pmatrix} A c_1^B & A c_2^B & \cdots & A c_n^B \end{pmatrix}$$

$$= \begin{pmatrix} r_1^A\, B \\ r_2^A\, B \\ r_n^A\, B \end{pmatrix}$$

Elementary matrices are matrics obtained performing 1 elementary operation on I

Performing elementary operations on a $n \times m$ matrix B corresponds to multiplying B by an elementary ($n \times n$) matrix

1) exchanging row i and row j

$$\begin{pmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \\ e_n \end{pmatrix} B$$

2) Multiplying row i by k

$$\begin{pmatrix} e_1 \\ \vdots \\ k e_i \\ \vdots \\ e_n \end{pmatrix} B$$

3) Replacing row i with row i + k row j

$$\begin{pmatrix} e_1 \\ \vdots \\ e_i + k e_j \\ \vdots \\ e_n \end{pmatrix} B$$

Ths: $Det(A \cdot B) = Det(A) \cdot Det(B)$

proof: If A is singuler then AB is singuler since the columns of AB are lineer combinations of columns of A (or by contrediction if AB is invertible then there is C s.t $(AB) \cdot C = I$ but then $A(BC) = I$ so A is invertible)

so $det(AB) = det(A) \cdot det(B) = 0$

If A is not singuler we can find elementery metrices. s.t $A = E_1 \cdots E_n I$ and we can prove by induction on n thet $det(E_1 \cdots E_n B) = det(E_1 \cdots E_n) \cdot det(B)$

① Base case

a) E_1 is obtained from I by swepping two rows then $E_1 B$ is the metrix obtained by swepping the same 2 rows in B

$\det(E_1 B) = -\det(B)$
$\det(E_1) \cdot \det(B) = -1 \cdot \det(B)$

b) E_1 is obtained by multiplying one row of I by $k (\neq 0)$
then $E_1 B$ is B with the same row multiplied by k
$\det(E_1 B) = k \det(B)$
$\det(E_1) \cdot \det(B) = k \cdot \det(B)$

c) E_1 is obtained by replacing row i of I with row i + k row j
then $E_1 B$ is the matrix obtained from B by replacing row i with row i + k row j

$\det(E_1 B) = \det(B)$
$\det(E_1) \cdot \det(B) = 1 \cdot \det(B)$

Induction step:

Assume $\det(E_1 \cdots E_n B) = \det(E_1 \cdots E_n) \det(B)$
then $\det(E_1 \cdots E\, E_{n+1} B) = \det(E_1) \det(E_2 \cdots E_{n+1} B)$
$= \det(E_1) \det(E_2 \cdots E_{n+1}) \det B = \det(E_1 E_2 \cdots E_{n+1}) \det B$
$= \det(A) \det(B)$

Corollery : $\mathrm{Det}(A^{-1}) = \dfrac{1}{\mathrm{Det}(A)}$

Th $\det(A^t) = \det(A)$

Hw problem

Quiz just compute det (3×3)

Compute el op on $I \rightsquigarrow A$

$\det A =$ eny c $\qquad$ $\det A = \pm 1$

$\det A =$ eny $c \neq 0$

$f(I) = 1$

if M is singuler then M^t is singular sof $\det(M^t) = 0$ then by the uniqueness th $\det A = \det(A^t)$