Lesson 14

Linear transformations

Change of coordinates

Textbook 2.5

Th if V is a finite dimensional vector space and $B_1 = \{v_1, \dots v_n\}$ is a basis for V then

$$\varphi \quad V \longrightarrow \mathbb{R}^n$$
$$\varphi(v) = [v]_B$$

is an isomorphism.

Proof: Let $e_1 = (1, \dots 0)$ $e_2 = (0, 1, \dots 0)$ $\dots$ in $\mathbb{R}^n$ and let $B_2 = \{e_1, e_2, \dots e_n\}$ then $\{\varphi(v_1), \varphi(v_2), \dots \varphi(v_n)\} = B_2$ is a basis for $\mathbb{R}^n$

Note: rembember isomorphisms map linearly independent sets to linearly independent sets, spanning sets to spanning sets, bases to bases

Th If $\dim V = n$ and $\dim W = m$
$\mathcal{L}(V, W) \approx M_{m \times n}(F)$ therefore
$\dim(\mathcal{L}(V, W)) = m \times n$

Proof: Fix bases $B_1 = \{v_1, v_2, \ldots v_n\}$ in V and $B_2 = \{w_1, w_2, \ldots w_m\}$ in W and consider $\varphi \quad \mathcal{L}(V, W) \to M_{m \times n}$

$$\varphi(T) = T_{B_1}^{B_2}$$

We want to show φ is an isomorphism.

1) It is a linear transformation proved in Lesson 11

2) It is 1-1 since $\varphi(T) = 0$ means $T_{B_1}^{B_2} = 0$ so $T = 0$

3) It is onto since given $M = (a_{ij})$ in $M_{m \times n}$ $M = \varphi(T) = T_{B_1}^{B_2}$ for the linear transformation T such that
$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m$
for $j = 1, \ldots n$

Def $\mathcal{L}(V, F)$ is called the dual space of V

often denoted by V^*

Matrix of change of basis

Def: $I \quad V \longrightarrow V$ is the linear transformation defined by $I(v) = v$.

Def Given bases B_1 and B_2 in V, we call $I_{B_1}^{B_2}$ the matrix of change of basis from B_1 to B_2. If $B_1 = v_1, v_2, \dots v_n$

$$I_{B_1}^{B_2} = \begin{bmatrix} [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ \vdots & & \vdots \end{bmatrix}$$

$$I_{B_1}^{B_2} [v]_{B_1} = [v]_{B_2}$$

Th: $$I_{B_2}^{B_1} = \left(I_{B_1}^{B_2}\right)^{-1}$$

Proof $$I_{B_2}^{B_1} \cdot I_{B_1}^{B_2} = I_{B_1}^{B_1} = I$$

Th: If $T \in \mathcal{L}(V)$ and B_1 and B_2 are bases for V then

$$T_{B_2}^{B_2} = I_{B_1}^{B_2} \circ T_{B_1}^{B_1} \circ I_{B_2}^{B_1}$$

Proof $\quad I_{B_1}^{B_2} \; T_{B_1}^{B_1} \; I_{B_2}^{B_1} \; [v]_{B_2} =$

$I_{B_1}^{B_2} \; T_{B_1}^{B_1} \; [v]_{B_1} =$

$I_{B_1}^{B_2} \; [Tv]_{B_1} =$

$[T(v)]_{B_2}$

So For the example in video 12

how are

$T_B^B \qquad T_B^{B_1} \qquad T_{B_1}^{B_1}$

related?

Hw problem: Find a basis for

$\mathcal{L}(R^2, R^3)$

① $\mathcal{L}(R^2, R^3) \simeq M_{3\times 2}$

Basis for $M_{3\times 2}$ is :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$M_5 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad M_6 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

② Fix the beses $B_1 = \{(1,0)\ (0,1)\}$
$B_2 = \{(1,0,0)\ (0,1,0),\ (0,0,1)\}$ in
R^3. Consider the isomorphism
$\varphi\ \mathcal{L}(R^2, R^3) \to M_{3\times 2}$ $\quad \varphi(T) = T_{B_1}^{B_2}$
find
linear transformations $T_1, T_2, T_3, T_4, T_5, T_6$
s.t $\quad \varphi(T_i) = M_i$

In other words you want to Think: if $M_1 = T_{B_1}^{B_2}$ what is T?

For example if

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = T_{B_1}^{B_2} = \begin{bmatrix} \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{B_2} & \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{B_2} \end{bmatrix}$$

what is T?

$$T\begin{pmatrix}0\\1\end{pmatrix} = 1\cdot(1\ 0\ 0) + 0\,(0\ 1\ 0) + 0\,(0\ 0\ 1)$$
$$= (1\ 0\ 0)$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = 0\,(1\ 0\ 0) + 0\,(0\ 1\ 0) +$$
$$+\,0\,(0\ 0\ 1) = (0, 0, 0)$$

so $T(x\ y) = T(x\,(1\ 0) + y\,(0\ 1))$

$= x\,T(1\ 0) + y\,T(0\ 1) =$

$= x\,(1\ 0\ 0) + y\,(0\ 0\ 0) = (x\ 0\ 0)$

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$$I^{B_2}_{B_1} = \begin{bmatrix} [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ \vdots & & \vdots \end{bmatrix}$$

$$I^{B_2}_{B_1} [v]_{B_1} = [v]_{B_2}$$

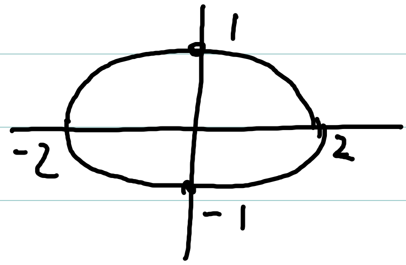
Th: $$I^{B_1}_{B_2} = \left(I^{B_2}_{B_1}\right)^{-1}$$

Proof $$I^{B_1}_{B_2} \cdot I^{B_2}_{B_1} = I^{B_1}_{B_1} = I$$

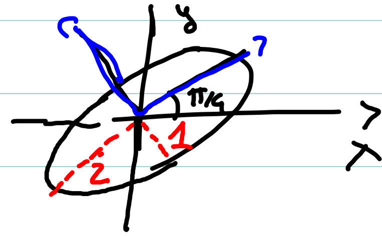
Th: If $T \in \mathcal{L}(V)$ and β_1 and β_2 are bases for V then

$$T_{\beta_2}^{\beta_2} = I_{\beta_1}^{\beta_2} \cdot T_{\beta_1}^{\beta_1} \cdot I_{\beta_2}^{\beta_1}$$

Application:



I went to rotate it counterclockwise of an angle of $\frac{\pi}{4}$



Find equation in x, y plane

$$\frac{x'^2}{2^2} + \frac{y'^2}{1} = 1$$

$(x' \; y')$ coordinates with respect to basis $B_2 =$

$\sqrt{2}$
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