Lesson 13

Inverses- Isomorphism

Textbook 2.4

Def: Given $T: V \longrightarrow W$, T^{-1} the inverse of T
(if it exists) is the function T^{-1} $W \to V$ s.t
$T(v) = w \Leftrightarrow T^{-1}(w) = v$ i.e $T \circ T^{-1} = I_W$ $T^{-1} \circ T = I_W$

Th: T^{-1} exists if and only if T is
one to one and onto.
In this case we say T is invertible.

Th: If $T\ V \to W$ is an invertible linear transformation then $T^{-1}\ W \to V$ is a linear transformation.

Th: If V is finite dim then $T: V \to W$ is invertible $\iff \dim R(T) = \dim V = \dim W$

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Th: Let V, W be finite dimensional

Let $T\ V \to W$ be a linear transformation and let B_1 and B_2 be ordered bases in V and W; then T is invertible iff $T_{B_1}^{B_2}$ is invertible, and $T^{-1}{}_{B_2}^{B_1} = \left(T_{B_1}^{B_2}\right)^{-1}$

Def If $T\ V \to W$ is an invertible linear transformation we call T an ISOMORPHISM and we say V and W are isomorphic, and we write $V \approx W$

Th Let $T\ V \to W$ be an isomorphism.

$v_1\ v_2 \cdots v_n$ are linearly independent in $V \iff$

$T(v_1)\ T(v_2) \cdots T(v_n)$ are linearly independent

in W.

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Th: If $T\ V \to W$ is an isomorphism, then $\{v_1, v_2, \cdots v_n\}$ spans V $\Leftrightarrow$ $\{T(v_1), T(v_2), \cdots T(v_n)\}$ sans W

Th If $T: V \to W$ is a linear transformation and $B_1 = \{v_1, v_2, \dots v_n\}$ is a basis for V and $B_2 = \{T(v_1)\ T(v_2) \cdots T(v_n)\}$ Then T is an isomorphism iff B_2 is a basis for W

Th if V is a finite dimensional vector space and $B_1 = \{v_1, \dots v_n\}$ is a basis for V then

$$\varphi \quad V \longrightarrow \mathbb{R}^n$$
$$\varphi(v) = [v]_B$$

is an isomorphism.

Proof: Let $e_1 = (1, \dots 0)$ $e_2 = (0, 1, \dots 0)$ $\dots$ in $\mathbb{R}^n$ and let $B_2 = \{e_1, e_2, \dots e_n\}$ then $\{\varphi(v_1), \varphi(v_2), \dots \varphi(v_n)\} = B_2$ is a basis for $\mathbb{R}^n$

$(0, 0, \cdot, \; , v,)$ are

Th If $\dim V = n$ and $\dim W = m$

$\mathcal{L}(V, W) \approx M_{m\times n}(F)$ therefore

$\dim(\mathcal{L}(V, W)) = m \times n$

Def $\mathcal{L}(V, F)$ is called the dual space of V

often denoted by V^*

Hw problem: Find a basis for

$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$

① $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \simeq M_{3\times 2}$

Basis for $M_{3\times 2}$ is :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$M_5 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad M_6 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

② Fix the bases $B_1 = \{(1,0)\ (0,1)\}$ $B_2 = \{(1,0,0)\ (0,1,0),\ (0,0,1)\}$ in $\mathbb{R}^3$. Consider the isomorphism φ in the previous slide, find linear transformations $T_1, T_2, T_3, T_4, T_5, T_6$ s.t $\varphi(T_i) = M_i$

In other words you want to
Think : if $M_1 = T_{B_1}^{B_2}$ what is T ?

For example if

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = T_{B_1}^{B_2} = \begin{bmatrix} \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{B_2} & \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{B_2} \end{bmatrix}$$

what is T ?

$$T\begin{pmatrix}0\\1\end{pmatrix} = 1\cdot(1\,0\,0) + 0\,(0\,1\,0) + 0\,(0\,0\,1)$$
$$= (1\,0\,0)$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = 0\,(1\,0\,0) + 0\,(0\,1\,0) +$$
$$+\,0\,(0\,0\,1) = (0, 0, 0)$$

so $T(x\; y) = T(x\,(1\,0) + y\,(0\,1))$

$= x\,T(1\,0) + y\,T(0\,1) =$

$= x\,(1\,0\,0) + y\,(0\,0\,0) = (x\;0\,0)$