Lesson 11

Algebra of linear transformations

Textbook 2.3

Algebra of linear transformations

Let V and W be vector spaces over F

Def: Let $\mathcal{L}(V,W)$ be the set of all linear transformations from V to W

Sometimes we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V,V)$.

Troughout this lesson, if we need to assume V, W are finite dimensional, then we assume $B_1 = \{v_1, v_2, \dots v_n\}$ is an ordered basis for V and $B_2 = \{w_1, w_2, \dots w_m\}$ is an ordered basis for W.

Our first goal is to define $+$ and scalar multiplication on $\mathcal{L}(V,W)$

Def Let $T \in \mathcal{L}(V,W)$, let $k \in F$ and
$kT : V \to W$ be defined by
$(kT)(v) = k\,T(v)$ then

Th: kT_1 is a linear transformation.

Proof:

Def: If $T_1, T_2 \in \mathcal{L}(V, W)$ let $T_1 + T_2 : V \to W$ be defined by $(T_1+T_2)(v) = T_1(v) + T_2(v)$

Th $T_1 + T_2 \quad V \to W$ is a linear transformation.

Hw problem

Th $\mathcal{L}(V\ W)$ is a vector space over F

1) $+$ is commutative i.e $T+S=S+T$ for all S, T in $\mathcal{L}(V,W)$

2) $+$ is associative, i.e $(A+B)+C = (A+B)+C$, for all A, B, C in $\mathcal{L}(V,W)$

3) $0: V \to W$, $0(v)=0$ is the "0 element in $\mathcal{L}(V,W)$ i.e $0+T=T+0$ for all T in $\mathcal{L}(V,W)$

4) Given $T \in \mathcal{L}(V,W)$ $-T$ is the transformation $(-T)\ V \to W$ $(-T)(v) = -T(v)$

5) $1 \cdot T = T$ for all T in $\mathcal{L}(V,W)$

6) $(a \cdot b)T = a \cdot (b \cdot T)$ for all a, b in F and T in $\mathcal{L}(V,W)$

7) $a(T+S) = aT + aS$ for all a in F and T, S in $\mathcal{L}(V,W)$

8) $(a+b)T = aT + bT$ for a, b in F, T in $\mathcal{L}(V,W)$

Idea: if $\dim V = n$ and $\dim W = m$ and we fix ordered bases B_1 in V and B_2 in W. Every $T \in \mathcal{L}(V, W)$ corresponds to a $m \times n$ matrix $T_{B_1}^{B_2}$

We want to argue that $\mathcal{L}(V, W)$ is "the same" as $M_{m \times n}(F)$.

The next theorem shows

$$\varphi : \mathcal{L}(V, W) \to M_{m \times n}(F)$$

$$\varphi(T) = T_{B_1}^{B_2}$$

is a linear transformation

Recall $[\ ]_{B_2} : W \to R^m$

$w \to [w]_{B_2}$

is also a linear transformation

Th If V and W are finite dimensional and B_1 is an ordered basis for V and B_2 is an ordered basis for W then

$$(T_1 + T_2)_{B_1}^{B_2} = {T_1}_{B_1}^{B_2} + {T_2}_{B_1}^{B_2}$$

and

$$(K T_1)_{B_1}^{B_2} = K \left({T_1}_{B_1}^{B_2} \right)$$

Proof:

$$\varphi(T) = T_{B_1}^{B_2}$$

is 1-1 and onto

Proof: suppose $B_1 = \{v_1 \dots v_n\}$
$B_2 = \{w_1 \dots w_m\}$

Th Let $T\ V \to W$, $S\ W \to U$ be linear tranformations; then ST (the composition of S and T) $V \to U$ is a linear transformation.

$ST(v) = S(T(v))$

Proof:

th Let V, W, U be finite dimensional and

Let $\beta_1, \beta_2, \beta_3$ be ordered bases for V, W, U

then $(ST)_{\beta_1}^{\beta_3} = S_{\beta_2}^{\beta_3} T_{\beta_1}^{\beta_2}$