Lesson 10

Linear transformations

Matrix representation

Textbook: 2.2

Def An ordered basis for a finite dimensional vector space V is a basis endowed with a specific order.

Def Given a vector space V, an ordered basis $\beta = \{v_1, \cdots, v_n\}$ for V and a vector $v \in V$ we define the coordinate vector of v relative to β, $[v]_\beta$ as the <u>unique</u> vector $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ s.t $v = d_1 v_1 + \cdots + d_n v_n$

Note: this allows us to think of a vector in a space of dimension n as an element of $\mathbb{R}^n$

Ex Take $P_3(\mathbb{R})$ $\quad B = \{1, x, x^2, x^3\}$

$[x^3 - x + 5]_\beta = (5, -1, 0, 3)$

Th Given a finite dimensional vector space V and an ordered basis $B_1 = \{v_1, v_2, \dots v_n\}$ for V

$$T \quad V \longrightarrow R^n$$

$$T(v) = [v]_{B_1}$$

is a linear transformation and it is injective and surjective.

Proof: suppose $v, w \in V$

$v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$

$w = h_1 v_1 + h_2 v_2 + \cdots + h_n v_n$

then $v + w = (k_1 + h_1) v_1 + \cdots + (h_n + k_n) v_n$

$\quad\quad d v = d k_1 v_1 + \cdots + d k_n v_n$

$T(v + w) = (k_1 + h_1, k_2 + h_2, \cdots k_n + h_n) =$

$\quad\quad (k_1 \cdots k_n) + (h_1 \cdots h_n) = T(v) + T(w)$

$T(d v) = (d k_1 \cdots d k_n) = d(k_1 \cdots k_n) = d T(v)$

T is surjective (onto): given any vector $(y_1 \cdots y_n)$ in R^n take $v = y_1 v_1 + \cdots + y_n v_n$

Then $T(v) = (y_1 \cdots y_n)$

T is injective (one to one): if $T(v) = T(w) = (x_1 \cdots x_n)$

Then $v = w = x_1 v_1 + \cdots + x_n v_n$

Th : Given a basis $B = \{v_1, v_2, \dots v_n\}$ in V and n vectors $\{w_1, \dots w_n\}$ in W

There is a unique linear transformation $T \; V \to W$ s.t $T(v_i) = w_i$

Proof :

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$T: R^3 \longrightarrow R^2$

$B_1 = \{e_1, e_2, e_3\}$ i.e $\{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

$B_2 = \{e_1, e_2\}$ i.e $\{(1,0)\ (0,1)\}$

The matrix of T is

$$M = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Then $T(v) = Mv$

why? If $S\ R^3 \longrightarrow R^3$ is defined by $Sv = Mv$

$M\begin{pmatrix}1\\0\\0\end{pmatrix}$ = first column of M so $Se_1 = Te_1$

$M\begin{pmatrix}0\\1\\0\end{pmatrix}$ = second column of M so $Se_2 = Te_2$

$M\begin{pmatrix}0\\0\\1\end{pmatrix}$ = third column of M so $Se_3 = Te_3$

Therefore $S = T$

Matrices of linear transformation

Given two finite dim spaces V and W with ordered bases $B_1 = \{v_1 \dots v_n\}$ and $B_2 = \{w_1 \dots w_m\}$ and a linear Transformation $T \quad V \longrightarrow W$ there exists a unique matrix $T_{B_1}^{B_2}$

s.t $\quad \underbrace{[T[v]]_{B_2}}_{\text{in } R^m} = T_{B_1}^{B_2} \underbrace{[v]_{B_1}}_{\text{in } R^n}$

Let's look at one example before we prove this theorem

Ex $\quad T: \underset{\approx R^3}{P_2(\mathbb{R})} \to \underset{\approx R^2}{P_1(\mathbb{R})}$

$T(p) = p'$

If we choose bases $B_1 = \{1, x, x^2\}$ in $P_2(\mathbb{R})$ and $\{1, x\}$ in $P_1(\mathbb{R})$ what is $T_{B_1}^{B_2}$?

IDEA: given a polynomial $p(x) = a \cdot 1 + b \cdot x + cx^2$ look at $[p]_{B_1} = (a, b, c) \in R^3$ instead, and

$T(p) = b + 2cx$ so $[T(p)]_{B_2} = (b, 2c) \in R^2$. Look at the linear transformation $S \; R^3 \to R^2$ instead $S(a,b,c) = (b, 2c)$

$\Big(S: (a,b,c) \to p(x) = a + bx + cx^2 \to T(p) = b + 2c \to (b, 2c) \Big)$
composition of linear transformations

$$S_{\{e_1\, e_2\, e_3\}}^{\{e_1\, e_2\}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = T_{B_1}^{B_2}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$$

Alternative way to find $T_{B_1}^{B_2}$

① Calculate $T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

② Find coordinates in $B_2 = \{1, x\}$

$0 = 0 \cdot 1 + 0 \cdot x$ so $[0]_{B_2} = (0, 0)$

$1 = 1 \cdot 1 + 0 \cdot x$ so $[1]_{B_2} = (1, 0)$

$2x = 0 \cdot 1 + 2 \cdot x$ so $[2x]_{B_2} = (0, 2)$

$$T_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

In general $T_{B_1}^{B_2} = \begin{bmatrix} [T(B_1)]_{B_2} & \cdots & [T(B_n)]_{B_2} \\ \downarrow & & \downarrow \end{bmatrix}$

$\{v_1, \dots, v_n\}$

Recall: $T \quad V \longrightarrow W \quad B_1, B_2$ are ordered bases in V and W. We want a matrix $T_{B_1}^{B_2}$ s.t $T_{B_1}^{B_2}[v]_{B_1} = [T(v)]_{B_2}$

Proof: $T_{B_1}^{B_2} = \begin{bmatrix} [T(v_1)]_{B_2} & \cdots & [T(v_n)]_{B_2} \\ \vdots & & \vdots \end{bmatrix}$

Def: $M_{m \times n}(F)$ is the set of all $m \times n$ matrices with entries in F.

Th: $M_{m\times n}(F)$ is a vector space of dimension $m \times n$

Proof: from hw 1 you know it is a vector space.

A basis is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}$$

$$\cdots \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$