Lesson 1

Vector spaces and subspaces.

Textbook : 1.1 , 1.2

Def: a field is a set S, together with operations $+, \cdot$ and special elements $0, 1$ $(0 \neq 1)$ such that

1) $+$ is commutative i.e $x+y = y+x$ for all x, y in S
2) $+$ is associative i.e $(x+y)+z = x+(y+z)$ for all x, y, z in S
3) 0 is the identity for $+$ i.e $x+0 = 0+x = x$ for all x in S
4) Every element x in S has an additive inverse $(-x)$ in S s.t $x+(-x) = (-x)+x = 0$
5) $\cdot$ is commutative
6) $\cdot$ is associative
7) 1 is the identity for $\cdot$, that is $1 \cdot x = x \cdot 1 = x$ $\forall x \in S$
8) Every $x \neq 0$ in S has a multiplicative inverse x^{-1}. $x^{-1} \in S$ s.t $x \cdot x^{-1} = x^{-1} \cdot x = 1$
9) Multiplication distributes over addition i.e $x(y+z) = x \cdot y + x \cdot z$ for all x, y, z in S

Ex: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_2$

Note: In this class when we talk about a field F we mean $\mathbb{R}$ or $\mathbb{C}$

Def: a Vector space over a field F is a set V togeter with operation $+$ (addition) and scaler multiplication s.t

1) $+$ is commotative i.e $v+w=w+v$ for all v, w in V
2) $+$ is associative, i.e $(v+w)+z=v+(w+z)$ for all v, w, z in V
3) There is an element in V, which we will denote as 0 s.t $v+0=0+v=v$ for all v in V
4) Every v in V has an additive inverse, which we will denote $(-v)$ s.t $v+(-v)=(-v)+v=0$
5) $1\cdot v=v$ for all v in V
6) $(a\cdot b)v=a\cdot(b\cdot v)$ for all a, b in F and v in V
7) $a(v+w)=av+aw$ for all a in F and v, w in V

8) $(a+b)v=av+bv$ for all a, b in F, v in V

Ex: $\mathbb{R}^n$ over $\mathbb{R}$

$$\underbrace{(x_1, x_2, \dots x_n)}_{v} + \underbrace{(y_1, y_2, \dots y_n)}_{w} = \underbrace{(x_1+y_1, \dots\dots, x_n+y_n)}_{v+w}$$

$$\lambda (x_1, x_2 \dots x_n) = (\lambda x_1, \lambda x_2 \dots\dots \lambda x_n)$$

Ex $\mathbb{C}^n$ over $\mathbb{C}$

Ex $\mathbb{C}^2$ over $\mathbb{R}$

Is it really different from $\mathbb{C}^2$ over $\mathbb{C}$?

Ex $P(F)$: the set of all polynomials with coefficient in F
with usual $+$, and multiplication by a constant

Ex R^∞ : the set of all real sequences with

$\{x_n\} + \{y_n\} = \{x_n + y_n\}$

$\lambda \{x_n\} = \{\lambda x_n\}$

Ex Given $S \subseteq R$, R^S : the set of all functions $f : S \to R$ with $(f+g)(x) = f(x) + g(x)$

$(\lambda f)(x) = \lambda \cdot f(x)$

Ex $C[0,1]$ the set of all real functions defined on $[0,1]$

Ex $M_{n\times m}[F]$ the set of all $n \times m$ matrices with coefficients in F with usual $+$ and scaler multiplication

Some "algebra" properties of Vector Spaces:

Th: cancellation law for vector addition: $v + z = w + z \Rightarrow v = w$ for all v, w, z in V.

Proof: assume $v + z = w + z$, then

$v = v + 0_v = v + (z + (-z)) = v + z + (-z) = w + z + (-z) = w$

Th: 0_v is unique, $0_F \cdot v = 0_v$ for all v in V, $a \cdot 0_v = 0_v$ for all a in F

Proof: suppose x in V has the property that $x + v = v + x = v$ for all v in V then

$x + 0_v = 0_v$ (x additive identity) but also

$x + 0_v = x$ (0_v additive identity) so $x = 0_v$

$0_F \cdot v = \underline{0_F v + 0_v} = (0_F + 0_F) v + 0_v = \underline{0_F v + 0_F v} + 0_v$

by using cancellation property $0_v = 0_F v + 0_v = 0_F v$

$a \cdot 0_v = \underline{a 0_v + 0_v} = a(0_v + 0_v) + 0_v = a \cdot 0_v + \underline{a \cdot 0_v + 0_v}$

so again by cancellation law $0_v = a \cdot 0_v + 0_v = a \cdot 0_v$

Some "algebra" properties of Vector Spaces:

Th: $(-1)\cdot v = -v$ for all v in V

Def: A subset S of a vector space V is a subspace of V (sometimes I will write $S \leq V$) if

1) $0 \in S$
2) $u, v \in S \Rightarrow u+v \in S$ for all u, v in S
3) $u \in S$, $\lambda \in F \Rightarrow \lambda u \in S$ for all $\lambda \in F$, $u \in S$

So a subspace S of V is itself a vector space with the same addition and scaler multiplication as V.