Lesson 9

Linear transformations

Null space , Range

Textbook: 2.1

Recall given a linear transformation

$T \quad V \to W$

$N(T) = \{ v \in V \ / \ T(v) = 0 \}$

$R(T) = \{ w \in W \ / \ \exists v \in V \ \ T(v) = w \}$

Def $\dim(N(T))$ = nullity of T

$\dim(R(T))$ = rank of T

when $N(T)$ and $R(T)$ are finite dimensional.

Th1: (dimension th)

Given $T \quad V \to W$, if V is finite dimensional then

$\dim V = \text{nullity}(T) + \text{rank}(T)$

Proof: Suppose $\dim(V) = n$

$\dim(N(T)) = k$, $\{v_1, \dots, v_k\}$ is a basis for $N(T)$ $\{w_1, \dots w_j\}$ is a is a basis for $R(T)$ and $w_i = T(b_i)$

Then we will show $B=\{v_1 \dots v_k \ b_1 \dots b_s\}$ is a basis for V

1) B is linearly independent: assume

$\alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 b_1 + \dots + \beta_s b_s = 0$

then $\alpha_1 v_1 + \dots + \alpha_k v_k = -\beta_1 b_1 - \dots - \beta_s b_s$

therefore $T(\underbrace{\alpha_1 v_1 + \dots + \alpha_k v_k}) = -\beta_1 T(b_1) \dots - \beta_s T(b_s)$

$\qquad\qquad\qquad\qquad\qquad\quad 0$

so $\beta_1 = \beta_2 \dots = \beta_s = 0$ and therefore

$\alpha_1 = \dots = \alpha_k = 0$

2) B spans V; let $v \in V$ and

let $T(v) = w$ then $w = \beta_1 w_1 + \dots + \beta_j w_j$

$= \beta_1 T(b_1) + \dots + \beta_s T(b_s) = T(\beta_1 b_1 + \dots + \beta_s b_s) \dots$

so $w = T(v) = T(\beta_1 b_1 + \dots + \beta_j b_j)$

so $T(v - (\beta_1 b_1 + \dots + \beta_s b_s)) = 0$

Therefore $v - (\beta_1 b_1 + \dots + \beta_s b_s) \in N(T)$

and so $v - (\beta_1 b_1 + \dots + \beta_s b_s) = \alpha_1 v_1 + \dots + \alpha_k v_k$

so finally $v = \alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 b_1 + \dots + \beta_s b_s$

This shows $k + j = n$

Ex1 : $T: F^n \to F^m$

$$T(x_1, \dots, x_n) = \left(\sum_{l=1}^{n} a_{1l} x_l, \ \dots \ \sum_{l=1}^{n} a_{ml} x_l\right)$$

$$T(x_1, \dots x_n) = \underset{m \times n}{M} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

nullity of T = dim $N(T)$

= nullity of M

rank of T = dim $R(T)$

= rank M

$$n = \text{nullity}(M) + \text{rank}(M)$$

In a system with n variables

free variables + # dependent variables $= n$

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Recall: a function $f\ X \to Y$
is injective (1-1) if $\forall x_1, x_2 \in X$
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Th 2: $T\ V \to W$
is injective iff $N(T) = \{0\}$ i.e if
$T(v) = T(0) \Rightarrow v = 0$

Proof Suppose that T is injective
and $T(v) = 0$, since $T(0) = 0$ we
must have $v = 0$, so $N(T) = \{0\}$
Assume now $N(T) = \{0\}$, and $T(v) = T(w)$.
Then $T(v) - T(w) = 0$ so $T(v-w) = 0$
so $v - w = 0$ so $v = w$.

Recall: a function $f\ X \to Y$ is
surjective (onto)
if $\forall y \in Y\ \exists x \in X\ y = f(x)$
that is $Y = \text{Range}(f)$.

Therefore: $T\ V \to W$ is surjective iff
$R(T) = W$

Th3: Assume V and W are finite dimensional vector spaces with $\dim V = \dim W$.
Let $T: V \to W$ be a linear transformation
then the following are equivalent

1) T is one to one
2) T is onto
3) $\operatorname{rank}(T) = \dim V$

Proof 1) $\Rightarrow$ 3): nullity $= 0$ so $\dim V = \operatorname{rank} T$
3) $\Rightarrow$ 2) $\operatorname{rank}(T) = \dim V$ implies $\operatorname{rank}(T) = \dim(W)$ so $R(T) = W$
2) $\Rightarrow$ 1) T is onto $\Rightarrow$ $\operatorname{rank} T = \dim W = \dim V$
so nullity $= 0$

The situation is different for infinite dim vector spaces

Ex 2: $T \quad P(\mathbb{R}) \to P(\mathbb{R})$

$$T(p) = p \cdot x$$

check it is a linear transformation

it is 1-1 not onto

$N(T) = \{0\}$

$R(T) = \{ q \in P(\mathbb{R}) \mid q(0) = 0 \}$

Ex 3: $T \quad \mathbb{R}^\infty \to \mathbb{R}^\infty$

$$T(x_1, x_2 \cdots) = x_2, x_3 \cdots\cdots$$

is onto but not 1-1

$T(0\,0 \cdots\cdots) = T(1\,0\,0 \cdots\cdots)$

$N(T) = \{ x, 0, 0 \cdots \mid x \in \mathbb{R} \}$

$R(T) = \mathbb{R}^\infty$

Suppose $T\ V \to W$ is a linear transformation and V is not finite dimensional. Can one of $N(T)$ and $R(T)$ be finite dimensional? Can both $N(T)$ and $R(T)$ be finite dimensional?