Lesson 8

Linear transformations

Null space , Range

Textbook: 2.1

Def : given two vector spaces V and W over F

A function $T\ V \to W$ is a linear transformation iff

$T(\lambda v) = \lambda T(v) \quad \forall \lambda \in F, \forall v \in V$

$T(v+w) = T(v)+T(w) \quad \forall v, w \in V$

Ex 1 $T\ P(\mathbb{R}) \to P(\mathbb{R})$

$T(p) = p'$

is a linear transformation since

$T(kp) = (kp)' = kp' = kT(p) \quad \forall k \in \mathbb{R}, p \in P(\mathbb{R})$

$T(p+q) = (p+q)' = p' + q' = T(p)+T(q) \quad \forall p, q \in P(\mathbb{R})$

Ex 2 $T\ F^\infty \to F^\infty$

$T(x_1, x_2, x_3 \cdots) = (x_2, x_3 \cdots \quad)$

Is a linear transformation since

$T(k\{x_n\}) = T(\{kx_n\}) = (kx_2, kx_3, kx_4 \cdots\cdots) = k\,T(\{x_n\})$

$T(\{x_n\}+\{y_n\}) = T(\{x_n+y_n\}) = (x_2+y_2, x_3+y_3 \cdots) = (x_2, x_3 \cdots\cdots)$

$+ (y_2, y_3, \cdots\cdots) = T(\{x_n\}) + T(\{y_n\})$

Ex 3 $T\ \mathbb{R}^3 \to \mathbb{R}^3$ projection on y, z plane

$T(x_1, x_2, x_3) = (0, x_2, x_3)$

is a linear transformation

Ex 4 $T\ F^\infty \to F^\infty$

$T((x_1, x_2, x_n \cdots)) = (0, x_1, x_2, x_n \cdots\cdots\cdots)$ is a linear transformation

Ex 5 T $\mathbb{R}^2 \to \mathbb{R}^2$ reflection

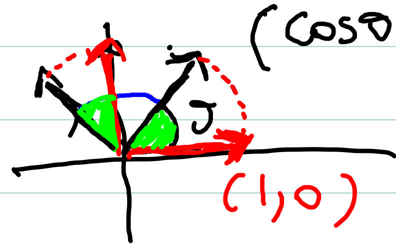
$T((x,y)) = (-x, y)$

Ex 6: T $\mathbb{R}^2 \to \mathbb{R}^2$ rotation

$$T((x,y)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$T((1,0)) = (\cos\theta, \sin\theta)$

$T((0,1)) = (-\sin\theta, \cos\theta)$

$(-\sin\theta, \cos\theta)$  $(\cos\theta, \sin\theta)$

Ex 7 : $T \quad R^n \to R^m$

$T(v) = Av$ where $A \in M_{mn}(R)$

Is a linear transformation since

$T(kv) = A(kv) = k(Av) = k\,T(v)$

$T(v+w) = A(v+w) = Av + Aw = T(v) + T(w)$

for all vectors $v, w \in R^n$ and scalars $k \in R$

Ex 8: $T \; R^3 \to R^3$

$T(x\,y\,z) = (x+1, 2y, z+x-y)$ NOT A LINEAR TRANSFORMATION

$T(1,0,0) = (2,0,1)$

$T(2,0,0) = (3,0,2) \neq 2T(1,0,0)$

Th 1: If $T \; V \to W$ is a linear transformation then $T(0) = 0$

Proof: $T(0) = T(0+0) = T(0) + T(0)$ so

$-T(0) + T(0) = -T(0) + T(0) + T(0)$ and

$0 = T(0)$

Def: Given $T\ V \to W$ The kernel of T (or null space) $N(T)$ is the set $\{v \in V / T(v) = 0\}$. The range of T, denoted $R(T)$ is the set $\{w \in W / w = T(v)$ for some v in $V\}$

Th2: $N(T) \leq V$, $R(T) \leq W$

Proof: if $v \in N(T)$, $T(v) = 0$ and $T(\alpha v) = \alpha T(v) = \alpha \cdot 0 = 0$
if $v, w \in N(T)$, $T(v) = T(w) = 0$ and $T(v+w) = T(v) + T(w) = 0$
Notice that if $N(T) \leq V$ then $0 \in N(T)$ so $T(0) = 0$

if $w \in R(T)$ then $T(v) = w$ for some $v \in V$ so $T(\alpha v) = \alpha T(v) = \alpha w$
so $\alpha w \in R(T)$; if $z \in R(T)$ then $T(u) = z$ for some $u \in V$
so $T(v+u) = T(v) + T(u) = w + u$

Th3: If V, W are vector spaces and $B = \{v_1, v_2, \dots v_n\}$ is a basis for V and $T\ V \to W$ is a linear transformation then $R(T) = \mathrm{Span}(T(v_1), \dots, T(v_n))$

Proof: if $w \in R(T)$ then $w = T(v)$ for some $v \in V$, and $v = k_1 v_1 + \cdots + k_n v_n$ for some scalars $k_1, k_2, \ldots k_n \in F$ so $w = T(k_1 v_1 + \cdots + k_n v_n) = k_1 T(v_1) + \cdots + k_n T(v_n)$

so $w \in \text{Span}(T(v_1) \cdots T(v_n))$ and therefore

$R(T) \leq \text{Span}(T(v_1), \cdots T(v_n))$

Vica-verse if $w \in \text{Span}(T(v_1) \cdots T(v_n))$ then

$w = k_1 T(v_1) + \cdots k_n T(v_n)$ for some $k_1, \ldots k_n \in F$

so $w = T(k_1 v_1 + \cdots + k_n v_n)$ so $w \in R(T)$ therefore

$\text{Span}(T(v_1) \cdots T(v_n)) \leq R(T)$

Ex 9 What are the null space and Range of

$T \quad P(R) \to P(R)$

$T(p) = p'$

$N(T) =$ constant polynomials

$R(T) =$ All polynomials

Ex 10: $T \quad F^\infty \to F^\infty$

$T(x_1, x_2 \cdots x_n \cdots) = (x_2, x_3, \cdots\cdots)$

$N(T) = \{ \{x_n\}, x_n = 0 \text{ for } n \geq 2 \}$

$R(T) = F^\infty$

Ex 11: $T: F^n \to F^m$

$$T(a_1, \ldots, a_n) = \underset{m \times n}{M} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$N(T)$ = corresponds to the solutions of the homogeneous system

$$M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$R(T)$ = Span (columns of M)