Lesson 7

Infinite dimensional

Textbook 1.7

Important facts about finite dimensional spaces

1. They have bases
2. Any independent set can be extended to a basis
3. Any spanning set contains a basis
4. All bases have the same number of elements

Examples $P(\mathbb{R})$

No finite bases, but $B = \{1, x, x^2, \dots\}$ is a basis

Example $\mathbb{R}^\infty$

$S = \{e_1, e_2, e_3, \dots e_n, \dots\}$

$e_i = 0, 0, \dots 0, \underset{i}{1}, 0, \dots$

Is S linearly independent? yes

Does it span $\mathbb{R}^\infty$ no, for example

$\{x_n\}$ $x_n = 1$ is not in $\mathrm{span}(S)$

Does $\mathbb{R}^\infty$ have a basis?

(not a countable one)

Axiom of choice and infinite dimensional vector spaces:

Axiom of Choice: given any family $\mathcal{F}$ of nonempty sets ($\mathcal{F} = \{S_x \mid x \in X\}$) there is a function $f: \mathcal{F} \to \bigcup_{x \in X} S_x$ s.t. $f(S) \in S$

Maximal priciple: Let $\mathcal{F}$ be a family of sets. If for every chain $C \cdots S_x \subseteq S_y \subseteq S_z \cdots$ of sets in $\mathcal{F}$ there exists a set in $\mathcal{F}$ that contains every set in C then $\mathcal{F}$ contains a maximal element, that is a set S that is not properly contained in any set of $\mathcal{F}$

Ex 1 of a family that does not have this property: $\mathcal{F}$ is a family of subsets of $\mathbb{R}$ $S \in \mathcal{F}$ if $S = (0, b)$ with $b > 0$ Consider the chain

$$(0,1) \subseteq (0, 1+\tfrac{1}{2}) \subseteq (0, 1+\tfrac{1}{2}+\tfrac{1}{4}) \subseteq \cdots$$ All included in $(0,3)$

but

$$(0,1) \subseteq (0,2) \subseteq \cdots (0,n) \subseteq \cdots$$

is a problem

Th: The axiom of choice and maximal principle are equivalent.

Th Let S be a linearly independent subset of a vector space V, then there exists a maximal linearly independent subset of V that contains S. Such maximal linearly independent set B is a basis for V.

Proof let $\mathcal{F}$ be the family of all linearly independent sets of V containing S. Given any chain C in $\mathcal{F}$
$\dots S_x \subset S_y \subset S_z \dots$ Take $T = \bigcup_{\text{all } S} S_x$ T is linearly independent since if $v_1 \dots v_n \in T$ then $v_1, v_2 \dots v_n$ must be in some set S in the chain C and S is linearly independent, so if $k_1 v_1 + k_2 v_2 + \dots k_n v_n = 0$ we must have $k_1 = k_2 = \dots = k_n = 0$
so by the maximal principle
$\mathcal{F}$ has a maximal set B. B is linear independent and it spans V since if $v \notin \mathrm{Span}(B)$ then $B \cup \{v\}$ would be linearly independent and therefore it would belong to $\mathcal{F}$.

Compare with proof for finite dimensional case: if S is an independent subset of some vector space V,

$S = S_1 = \{v_1, v_2, \ldots v_k\}$ we can keep adding vectors until we have a basis.

$$S = S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m = B_1$$
$$\subseteq T_2 \subseteq \cdots \subseteq T_m = B_2$$

Infinite dimensional

$$S = S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots \subseteq S_\omega \subseteq S_{\omega+1} \subseteq \cdots \quad B_1$$

What maximal principle says is that this process must terminate

Th Let V be a vector space and let S span V then S contains a basis for V

Proof: Consider $\mathcal{F} = \{T \subseteq S$

T is linearly independent$\}$

if $\cdots S_x \subseteq S_y \subseteq S_z \subseteq \cdots$ is a chain of elements of $\mathcal{F}$ then $\bigcup S_x$ is a subset of S and it is linearly independent (same argument as in the previous proof.) so it is in $\mathcal{F}$

By the maximal principle $\mathcal{F}$ contains a maximal element B. B is independent since it is in $\mathcal{F}$ and it must span V since if it does not and $v = k_1 s_1 + \cdots + k_m s_n$ for $k_i \in F$ $s_i \in S$ and $v \notin span(\beta)$ then some $s_i \notin Span(B)$ so $B \cup \{s_i\}$ is an independent subset of S bigger than B. Impossible.