Lesson 6

1.6

Dimension of subspaces

Th if $W \leq V$ and V is finite dimensional then W is finite dimensional and $\dim W \leq \dim V$. If $\dim W = \dim V$ then $W = V$

Proof let $\dim V = n$

if $W = \{0\}$ $\dim W = 0 \leq n$

if $W \neq \{0\}$ there is $U \neq 0$ in W we can build an independent set $S \subseteq W$ starting with U and adding vectors if we can (that is if we find a vector $y \in W$ $y \notin \text{Span}(S)$ then we can add it) but S is also an independent subset of V so it cannot have more than n elements Therefore this process has to stop after at most n steps and when it does, S is a maximal independent subset of W, therefore it must span W and be a basis.

(maximal means not included in any other independent subset of W)

If S has n elements then it is also a basis for V so

$W = \text{span}(S) = V$

Th: If U_1 and U_2 are subspaces of a finite dimensional vector space then
$\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$
$\dim(U_1 \oplus U_2) = \dim U_1 + \dim(U_2)$

Proof: Let $\{x_1, \dots x_m\}$ be a basis for $U_1 \cap U_2$ (possibly an empty set if $U_1 \cap U_2 = \{0\}$)
this is a linearly independent set in U_1, so it can be extended to a basis $x_1 \dots x_m, v_1 \cdots v_j$ of U_1 and similarly it can be extended to a basis $x_1, \dots x_m, w_1 \dots w_k$ of U_2
We will show that $x_1, \dots x_m, v_1, \dots, v_j, w_1, \dots, w_k$ is a basis for $U_1 + U_2$:

Claim 1:

$x_1, \dots, x_m\ v_1 \dots v_j\ w_1 \dots w_k$ are linearly independent

Suppose $a_1 x_1 + \dots + a_m x_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + \dots + c_k w_k = 0$ $(*)$

then $c_1 w_1 + \dots + c_k w_k = -a_1 x_1 \dots - a_m x_m - b_1 v_1 - \dots - b_j$

so $c_1 w_1 + \dots + c_k w_k \in U_1$ and therefore to $U_1 \cap U_2$

so $c_1 w_1 + \dots + c_k w_k = d_1 x_1 + \dots + d_m x_m$

and therefore going back to $(*)$

$a_1 x_1 + \dots + a_m x_m + b_1 v_1 + \dots + b_j v_j + d_1 x_1 + \dots + d_m x_m = 0$

implies $a_1 + d_1 = \dots = a_m + d_m = b_1 = \dots = b_j = 0$ so again $(*)$

becomes $a_1 x_1 + \quad + a_m x_m + c_1 w_1 + \dots + c_k w_k = 0$ therefore

$a_1 = \dots = a_m = b_1 = \dots = b_j = 0$

Claim 2

$x_1 \dots x_m\ v_1 \dots v_j\ w_1 \dots w_k$ span $U_1 + U_2$:

obvious since

$\{x_1 \dots x_m\ v_1 \dots v_j\}$ spans U_1 and

$\{x_1 \dots x_m\ w_1 \dots w_k\}$ spans U_2

Th $V = U_1 + U_2 + \cdots + U_n$ is a direct sum iff
$\dim V = \dim U_1 + \dim U_2 + \cdots + \dim U_n$

Proof : Let $B_j = b_1^j \dots b_{k_j}^j$ be a basis for U_j
clearly $B = B_1 \cup \cdots \cup B_n$ spans V
Assume the sum is direct, we want to show B is linearly independent
If it is it is a basis for V,
therefore $\dim V = \#$ vectors in B =
$\dim U_1 + \dim U_2 + \cdots + \dim U_n$: suppose

$$\underbrace{(a_1^1 b_1^1 + \cdots + a_{k_1}^1 b_{k_1}^1)}_{w_1 \in U_1} + \underbrace{(a_1^2 b_1^2 + \cdots + a_{k_2}^2 b_{k_2}^2)}_{w_2 \in U_2}$$

$$+ \cdots + \underbrace{(a_1^n b_1^n + \cdots a_{n_k}^n b_{n_k}^n)}_{w_n \in U_n} = 0$$

since the sum is direct

$$(w_1 + w_2 + \cdots + w_n = 0 \implies w_1 = w_2 = \cdots = w_n = 0)$$

and if $w_1 = a_1^1 b_1^1 + \cdots + a_{k_1}^1 b_{k_1}^1 = 0$
then $a_1^1 = a_2^1 = \cdots = a_{k_1}^1 = 0$
(because $b_1^1, b_2^1, \dots b_{k_1}^1$ are linearly independent)

Similery all other scalers are equal to 0 so B is independent.

Now assume $\dim V = \dim U_1 + \cdots + \dim U_n$

our goal is to show the sum is direct.

B is a basis for V since it spans V and has $\dim V$ elements. Therefore B is an independent set.

Assume $(*)$ $0 = w_1 + w_2 + \cdots + w_n$ with $w_j \in U_j$

then w_j can be witten as lineer combination of vectors in B_j:

$w_j = a^j_1 b^j_1 + a^j_2 b^j_2 + \cdots + a^j_{k_j} b^j_{k_j}$

and $(*)$ becomes

$$\underbrace{(a^1_1 b^1_1 + \cdots + a^1_{k_1} b^1_{k_1})}_{w_1 \in U_1} + \underbrace{(a^2_1 b^2_1 + \cdots + a^2_{k_2} b^2_{k_2})}_{w_2 \in U_2}$$

$$+ \cdots + \underbrace{(a^n_1 b^n_1 + \cdots a^n_{n_k} b^n_{n_k})}_{w_n \in U_n} = 0$$

Since B is linearly independent all scalers a^j_i must be 0 so

$w_1 = w_2 = \cdots = w_n = 0$

Th: suppose V is finite dimensional and $W \leq V$
then there exists $U \leq V$. s.t $V = U \oplus W$

Proof: Let $\{w_1 \dots w_k\}$ be a basis for W
extend it to a basis $\{w_1 \dots w_k, v_1 \dots v_m\}$
for V. Let $U = \text{span}\{v_1 \dots v_m\}$
$U \cap W = 0$: if $u \in U \cap W$ then $u = \lambda_1 v_1 + \dots + \lambda_m v_m = \alpha_1 w_1 + \dots + \alpha_k w_k$
and therefore $\lambda_1 v_1 + \dots + \lambda_m v_m - \alpha_1 w_1 - \dots - \alpha_k w_k = 0$
so $\lambda_1 = \dots = \lambda_m = \alpha_1 = \dots = \alpha_k = 0$ so $u = 0$
$\dim(U \oplus W) = \dim U + \dim W = k + m = \dim V$
so $U \oplus W = V$.