Lesson 5

Bases and dimension

Textbook: 1.6

Def: $B \subseteq V$ is a basis for V if
B is linearly independent and spans V

Note: We say ϕ is a basis for $V = \{0\}$

Th: Let $S = \{v_1, \dots, v_n\}$ span $V \neq \phi$
then S contains a basis for V

Proof: Consider the list $v_1, \dots, v_n$.
Construct B as follows

Step 1 put v_1 in $B \iff v_1 \neq 0$

step j put v_j in $B \iff v_j$ not in span(B).

End after n steps

At each step j B is linearly independent. Any discarded vector can be written as a linear combination of vectors in B so B still spans V

Th: Every finitely generated vector space has a finite basis.

Th: If V is finitely generated any finite independent set of vectors $v_1 \cdots v_k$ can be extended to a basis $v_1 \cdots v_k\ w_1 \cdots w_m$.

Proof: Let $s_1 \cdots s_n$ be a spanning set for V then $v_1 \cdots v_k\ s_1 \cdots s_n$ is still a spanning set and we can use the algorithm in the proof of the previous theorem.

If V is finitely generated

Th 1: Any finite spanning set S for V contains a basis for V

Th 2 Any finite independent set $S \subseteq V$ is contained in a basis of V.

Th (Replacement): Let V be a vector space, $V = \text{span}\{v_1, v_2, \cdots v_n\}$ and let $\{w_1, \dots w_m\} \subseteq V$ be linearly independent. Then $m \leq n$ and there exists $n-m$ vectors $v_{i_1} \dots v_{i_{n-m}}$ in $\{v_1 \cdots v_n\}$ such that the set $S = \{w_1 \dots w_m\} \cup \{v_{i_1} \dots v_{i_{n-m}}\}$ spans V.

Proof: By induction on m

If $m = 0$, clearly $m \leq n$; take $S = \{v_1 \dots v_n\}$

Assume now the theorem is true for $m \geq 0$ and let $\{w_1, \dots w_m\ w_{m+1}\}$ be linearly independent

Consider $L = \{w_1 \dots w_m\}$ then $m \leq n$ and there is $S = \{w_1 \dots w_m\ v_{i_1}, \dots w_{i_{n-m}}\}$ s.t $V = \text{span}(S)$

so $w_{m+1} = a_1 w_1 + \dots + a_m w_m + b_1 v_{i_1} + \dots + b_j v_{i_j} + \quad b_{i_{n-m}} v_{i_{n-m}}$

at least one b, say b_j is non zero

(because $w_1, w_2, \dots w_m, w_{m+1}$ are linearly independent)

So $v_{i_j} = W_{m+1} \quad -\frac{a_1}{b_j} w_1 \dots - \frac{a_m}{b_j} w_m - \frac{b_1}{b_j} v_1 \dots - \frac{b_{i_{n-m}}}{b_j} \ v_{n-m}$

Therefore $\{w_1 \dots w_{m+1}\} \cup \{v_{i_1} \dots v_{i_{j-1}}\ v_{i_{j+1}} \dots v_{i_{n-m}}\}$ spans V and there are $n-m-1 = n-(m+1)$ vectors in the second set, so $n \geq m+1$.

Th If V has one finite basis all of its bases have the same number of elements

Proof: Let $B_1 = \{v_1 \cdots v_n\}$ and $B_2 = \{w_1 \cdots w_m\}$ be two bases for V

B_2 is linearly independent and B_1 spans V so $m \leq n$. Viceversa B_2 spans V and B_1 is linearly independent so $n \leq m$.

it follows that $m = n$

Def: If V has a finite basis with n elements we say the dimension of V is n and we write $\dim V = n$.

Note: $\dim \{0\} = 0$

Th: $S = \{u_1, u_2, \dots, u_n\}$ is a basis for V iff each vector in V can be expressed in a unique way as linear combination of vectors of S.

Proof: Assume S is a basis for V, then every $v \in V$ can be written as a linear combination

$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$. Assume that

$v = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n$ as well. Then

$$(\alpha_1 - \beta_1) u_1 + (\alpha_2 - \beta_2) u_2 + \dots + (\alpha_n - \beta_n) u_n = 0$$

since S is independent $\alpha_i = \beta_i$ for all i

Now assume each vector can be written as a linear combination of elements of S, in a unique way then S spans V and 0 can only be written as $0 \cdot u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_n$ so S is independent.

Def If $B = \{b_1, b_2, \dots b_n\}$ is a basis for V and $v \in V$ $v = k_1 b_1 + \dots + k_n b_n$ we will denote $(k_1, k_2, \dots k_n)$ by $[v]_B$ and call this vector the vector of coordinates of v with respect to B.

Ex 1 $\quad \text{Dim } \mathbb{R}^n = n$

Basis $(1\,0\,0\cdots 0)\ (0\,1\,0\cdots 0)\ \cdots\ (0\cdots 0\,1)$

Ex: $\quad \text{Dim } \mathbb{C}^n = n$

Basis $(1\,0\,0\cdots 0)\ (0\,1\,0\cdots 0)\ \cdots\ (0\cdots 0\,1)$

Ex $\quad \text{Dim } M_{m\times n}(F) = m\cdot n$

Basis

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & & 0 \end{bmatrix} \quad \cdots$$

$$\cdots \quad \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

Ex $P_n(\mathbb{R})$ has dimension $n+1$

$\{1, x, x^2, \dots x^n\}$ is a basis

Ex $P(\mathbb{R})$ is not finite dimensional

$B = \{x^n \mid n \in \mathbb{Z}, n \geq 0\} = \{1, x, x^2, \dots\}$

is a basis for $P(\mathbb{R})$

Th Let $\dim V = n$ then

1) Any spanning subset S of V with n elements is a basis.
2) Any independent subset T of V with n elements is a basis.

Proof:

1) S must contain a subset β which is a basis for V, so β has n elements therefore $\beta = S$
2) T can be extended to a basis β for V with n elements so $T = \beta$

Note: If $\dim(V) = n$ any independent subset has $\leq n$ elements, any spanning set has $\geq n$ elements