Lesson 4

Spanning sets, linearly independent sets

Textbook: 1.4, 1.5

Def: Given $v_1, \dots, v_m \in V$, $\alpha_1, \alpha_2, \dots \alpha_m \in F$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ is called a linear combination of

$v_1 \cdots v_m$.

Def: Given $S \subseteq V$ span(S) is the set containing

all linear combinations of vectors of S.

Ex $\text{Span}((1 0)) = \{(x, 0)\}$

Ex $\text{Span}(\{(1 0 0), (0 1 0)\}) = \{(\alpha_1, \alpha_2, 0)\}$

By convention $\text{span}(\emptyset) = \{0\}$

Th: If $S \subseteq V$, Span(S) is the smallest subspea of V containing S.

Proof: obvious if $S = \emptyset$, if $S \neq \emptyset$

: 1) If $v \in S$ then $0 \cdot v = 0 \in S$

2) if $u = \alpha_1 v_1 + \cdots + \alpha_m v_m$ and

$w = \alpha_1 w_1 + \cdots + \alpha_k w_k$ are in Span (S)

Then $u + w = \alpha_1 v_1 + \cdots + \alpha_n v_m + \alpha_1 w_1 + \cdots + \alpha_s w_s$ is in Span (S)

3) if $u = \alpha_1 v_1 + \cdots + \alpha_m v_m$ is in Span (S)

then $\alpha u = (\alpha \cdot \alpha_1) v_1 + \cdots + (\alpha \cdot \alpha_m) v_m$ is in Span(S)

It is clear than any subspea containing S must contain span(S)

Ex: $V = P(R)$. span($\{x+1, x-1\}$) consists of all polynomials of the form $a(x+1) + b(x-1) = (a+b)x + a - b$, that is all linear polynomials since given $Ax + B$ we can always find a, b s.t

$A = a + b$

$B = a - b$

Def: Given $S \subseteq V$ we say S generates (or spans) V if $V = \text{span}(S)$

Def A vector space V is finite dimensional if there is a finite $S \subseteq V$ that spans it.

Ex $P(\mathbb{R})$ is not finite dimensional
Given any finite $S = \{p_1, \ldots p_k\}$
Let $n = \max_i \deg(p_i)$ Then $\text{span}(S)$

only contains polynomials of degree $\leq n$

Let $S = \{x^n \mid n \in \mathbb{N}\}$ then $P(\mathbb{R}) = \text{span}(S)$

Def: $v_1, \ldots, v_m \in V$ are linearly independent if $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$ implies $a_1 = a_2 \cdots = a_m = 0$. vectors that are not linearly independent are called linearly dependent.

Def $S \subseteq V$ is linearly independent if for every finite list $v_1, \ldots, v_m$ of vectors in S $v_1, \ldots, v_m$ are linearly independent.

Ex Two non zero vectors v and w are dependent iff there is $d \in F$ $v = dw$

proof: if v, w are dependent then there are $d_1, d_2 \in F$ not both equal to 0 s.t $d_1 v + d_2 w = 0$
If $d_1 = 0$ then $d_2 w = 0$ would imply $w = 0$, so $d_1 \neq 0$
and $v = -\frac{d_2}{d_1} w$.
If $v = dw$ then $1 \cdot v - dw = 0$
so v and w are dependent.

Ex: $\{0\}$ is linearly dependent

Ex:

Are
$$v_1 = (1, 1, -1, 0),$$
$$v_2 = (2, 3, 0, 1),$$
$$v_3 = (1, 2, -3, -1)$$
$$v_4 = (1, 1, 1, 1)$$

linearly independent ?

Consider

$$\lambda_1(1,1,-1,0) + \lambda_2(2,3,0,1) + \lambda_3(1,2,-3,-1) + \lambda_4(1,1,1,1) = (0,0,0,0) \quad ?$$

The vectors are linearly independent iff the system below has only the trivial solution :

$$\begin{aligned} \lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 &= 0 \\ \lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 &= 0 \\ -\lambda_1 + - 3\lambda_3 + \lambda_4 &= 0 \\ \lambda_2 - \lambda_3 + \lambda_4 &= 0 \end{aligned}$$

or

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ -1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ -1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

want this system to only have one solution $(0, 0, 0, 0)$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are free variables, system has infinitely many solutions, so v_1, v_2, v_3, v_4 are not independent. Pivot columns are independent: $(1,1,-1,0)$ $(2,3,0,1)$ $(1,1,1,1)$ and $\text{span}(v_1, v_2, v_3, v_4) = \text{span}(v_1, v_2, v_4)$

Proof: If we remove the columns that correspond to free variables we obtain a system that has only the 0 solution.

The span is the same since I can find sol

$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$ with $x_3 \neq 0$

so $v_3 = -\frac{x_1}{x_3} v_1 - \frac{x_2}{x_3} v_2 - \frac{x_4}{x_3} v_4$

Th: Let S be a linearly independent subset of V and let $v \in V$, $v \notin S$ then $S \cup \{v\}$ is linearly dependent iff $v \in Span(S)$

Proof: assume $v \in Span(S)$ then
$v = \alpha_1 w_1 + \cdots + \alpha_n w_n$ for some scalars $\alpha_1 \ldots \alpha_n$
and vectors $w_1, \ldots, w_n$ in S therefore
$\alpha_1 w_1 + \cdots + \alpha_n w_n - 1 \cdot v = 0$ shows $\{w_1 \ldots w_n, v\}$ is dependent
Vice-verse if $\{w_1 \ldots w_n, v\}$ is dependent we
must have $\alpha_1 w_1 + \cdots + \alpha_n w_n + \alpha v = 0$ for some
vectors $w_1 \ldots w_n \in S$, and not all scalers equal to 0
If $\alpha = 0$ then $\alpha_1 w_1 + \cdots + \alpha_n w_n = 0$ implies (since S is independent) that $\alpha_1 = \cdots = \alpha_n = 0$, therefore $\alpha \neq 0$
so $v = -\frac{\alpha_1}{\alpha} w_1 \cdots - \frac{\alpha_n}{\alpha} w_n$

Th If $v \in Span(S)$ then $Span(S \cup \{v\}) = Span(S)$

Proof since $v = \alpha_1 v_1 + \cdots \alpha_n v_n$ any linear combination of elements of S and v can be written as a linear combination of elements of S