Lesson 3

Sums, direct sums

1.3

Def: If S_1 and S_2 are subspaces of V the sum S_1+S_2 is the set

$$S_1 + S_2 = \{v+w \mid v \in S_1, w \in S_2\}$$

Exemple $\{(x\ 0) \mid x \in R\} + \{(0\ y) \mid y \in R\} = R^2$

since any vector (x, y) in R^2 can be written as a sum

$(x\ y) = (x, 0) + (0, y)$

For the rest of these notes if I write S_1+S_2 I will assume S_1 and S_2 are subspaces of some vector space V.

Th S_1+S_2 is the smallest subspace of V that contains S_1 and S_2

Proof: First we shall prove $S_1+S_2 \leq V$:

1) $0_V \in S_1+S_2$ since $0_V = 0_V + 0_V$ and $0_V \in S_1$, $0_V \in S_2$

2) If U_1 and $U_2 \in S_1+S_2$ then $U_1 = V_1 + w_1$, $U_2 = V_2 + W_2$ with $v_1, v_2 \in S_1$, $w_1, w_2 \in S_2$ therefore
$U_1 + U_2 = (V_1+V_2) + (w_1+w_2) \in S_1+S_2$

3) If $U \in S_1+S_2$ then $U = V + W$ with $V \in S_1$, $w \in S_2$ and $\lambda U = \lambda V + \lambda w \in S_1+S_2$ so $\lambda U \in S_1+S_2$

Then we can notice that any subspace W of V containing S_1 and S_2 has to contain S_1+S_2, because W is closed under addition, therefore S_1+S_2 is the smallest subspace of V containing S_1 and S_2

Note: in the video I gave a slightly different proof.

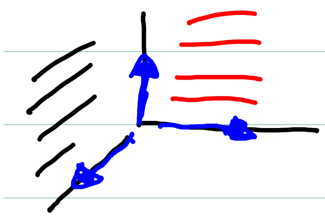
Note: In a similar way we can define the sum of k subspaces of V

$S_1 + S_2 + \cdots + S_k = \{ v_1 + v_2 + \cdots + v_k \mid v_i \in S_i \}$

it is still the smallest subspace containing $S_1 \cdots S_k$.

Example $S_1 = xz$ plane in $\mathbb{R}^3$ $\quad S_2$ yz plane in $\mathbb{R}^3$

$S_1 = \{ (x\ 0\ z) \mid x, z \in \mathbb{R} \}$ $\quad S_2 = \{ (0\ y\ z) \mid y, z \in \mathbb{R} \}$



$S_1 \cap S_2 = z$ axis

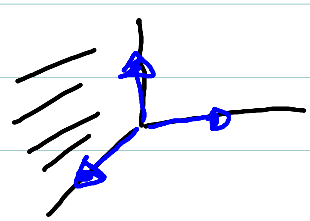
$S_1 + S_2 = \mathbb{R}^3$

$(1,1,1) = (1,0,1) + (0,1,0)$ or

$(1,1,1) = (1,0,0) + (0,1,1)$

On the other hand if $S_3 = \{ (0\ y\ 0) \mid y \in \mathbb{R} \}$ y axis

what is $S_1 + S_3$? $\quad S_1 + S_3 = \mathbb{R}^3$



s.

$(1\ 1\ 1) = (1\ 0\ 1) + (0\ 1\ 0)$ no other possibilitie

Def The Sum $S_1 + S_2 \cdots + S_n$ is called direct and we shall write $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ if every vector in it can be written in only one way as sum of vectors of S_i.

Th: $U_1 + U_2 \cdots + U_n$ is a direct sum iff $w_1 + w_2 + \cdots + w_n = 0_V$ with $w_i \in U_i$ implies $w_i = 0_V$ for $1 \le i \le n$

Proof $\Rightarrow$: Assume $U_1 + U_2 + \cdots + U_n$ is a direct sum, then by definition any vector including 0_V can be written in only one way as sum of vectors in U_i.

$\Leftarrow$ assume $w_1 + w_2 + \cdots + w_n = 0$, with $w_i \in W_i$ implies $w_i = 0_V$ for $1 \le i \le n$

and $v = v_1 + v_2 + \cdots + v_n = u_1 + u_2 + \cdots + u_n$ with $v_i \in U_i$, $u_i \in U_i$ then

$(v_1 - u_1) + (v_2 - u_2) + \cdots + (v_n - u_n) = 0_V$ therefore $v_i = u_i$ so each v can be written in only one way as sum of vectors in U_i.

Th: $U_1 + U_2$ is direct sum $\Leftrightarrow U_1 \cap U_2 = \{0_V\}$

Proof: $\Rightarrow$ suppose $U_1 + U_2$ is a direct sum and $w \in U_1 \cap U_2$ then $-w \in U_1 \cap U_2$

$w + (-w) = 0_V$ therefore $w = 0_V$

$\Leftarrow$ assume $U_1 \cap U_2 = \{0_V\}$

and $0_V = v + w$ with $v \in U_1$, $w \in U_2$

then $v = -w$ so $v \in U_1 \cap U_2$

so $v = w = 0_V$

Example: $f\ \mathbb{R} \to \mathbb{R}$ is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$. Let

$E = \{\text{Even functions } \mathbb{R} \to \mathbb{R}\}$ $\quad E \leq \mathbb{R}^{\mathbb{R}}$

Proof

1) The constant 0 function is even since $0(x) = 0$ and $0(-x) = 0$ for all x in $\mathbb{R}$, so $0(x) = 0(-x)$

2) If f and g are even then
$(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$ so $f+g$ is even

3) If f is even and $\lambda \in \mathbb{R}$ then
$(\lambda f)(x) = \lambda \cdot f(x) = \lambda f(-x) = (\lambda f)(-x)$
so λf is even.

$O = \{\text{odd functions } \mathbb{R} \to \mathbb{R}\}$ $\quad O \leq \mathbb{R}^{\mathbb{R}}$

Proof is similar as for E.

$\mathbb{R}^{\mathbb{R}} = E \oplus O$

Proof

$$f = \frac{1}{2}\underbrace{(f(x) + f(-x))}_{\text{EVEN}} + \frac{1}{2}\underbrace{(f(x) - f(-x))}_{\text{ODD}}$$

so $\mathbb{R}^{\mathbb{R}} = E + O$

$E \cap O = \{0\}$ since if $f \in \mathbb{R}^{\mathbb{R}}$ is both even and odd, for any x $f(x) = f(-x) = -f(x)$ so $2f(x) = 0$ so $f(x) = 0$.

Ex Any matrix $M \in M_{n\times n}(\mathbb{R})$ can be written in the form

$$A = \underbrace{\frac{1}{2}(A + A^t)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^t)}_{\text{Skew symmetric}}$$

$$M_{n\times n}(\mathbb{R}) = \text{Symmetric} \oplus \text{Skew-Symmetric}$$

Note: when considering $U_1 + \cdots + U_n$ it is not sufficient to have $U_i \cap U_j = \{0\}$ to ensure the sum is direct

Ex: $U_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

$U_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}$

$U_3 = \{(0, y, y) \mid y \in \mathbb{R}\}$

$(0, -1, 0) + (0, 0, -1) + (0, 1, 1) = (0, 0, 0)$

so the sum is not direct but

$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$