Lesson 24

Orthogonal Complement

6.2

Def Let S be a nonempty subset of an inner product space V. The orthogonal complement of S

$S^\perp = \{ v \in V, \langle v, s \rangle 0 \text{ for all } s \in S \}$

Th 1: $S^\perp \leq V$

$0 \in S^\perp$ since $\langle 0, v \rangle$ for any v in V

Assume $v, w \in S^\perp$ then

$\langle v + w \ \ s \rangle = \langle v \ \ s \rangle + \langle w \ \ s \rangle = 0$

therefore $v + w$ is in S

$\langle cv \ \ s \rangle = c \langle v \ \ s \rangle = 0$

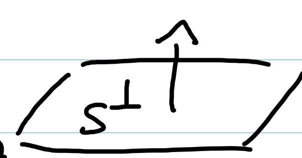
therefore cv is in S

Example $V = R^3$ $S = \{ (1, 1, 2) \}$

$(x, y, z) \perp (1, 1, 2)$ iff $\langle (x y z), (1,1,2) \rangle = 0$

iff $x + y + 2z = 0$

$S^\perp = \{ (x \ y \ z) \mid x + y + 2z = 0 \}$



$= \text{span}\{ (1, -1, 0) \ (-2, 0, 1) \}$

Th 2: Suppose V is an inner product space and Suppose U is a finite dimensional subspace of V, then $V = U \oplus U^{\perp}$

Proof:

Let $e_1 \cdots e_m$ be an orthonormal basis for U and let $v \in V$, then

$$v = \underbrace{\langle v\ e_1\rangle e_1 + \cdots + \langle v\ e_m\rangle e_m}_{v_1} + \underbrace{v - \langle v e_1\rangle e_1 \cdots - \langle v e_m\rangle e_m}_{v_2}$$

$v_1 \in U$, $v_2 \in U^{\perp}$ since $\langle v_2\ e_j\rangle =$
$= \langle v\ e_j\rangle - \langle v\ e_j\rangle \overset{1}{\langle e_j\ e_j\rangle} = 0$, therefore for any w in U if $w = a_1 e_1 + \cdots + a_m e_m$ then
$\langle v_2\ w\rangle = \sum_{i=1}^{m} \bar{a}_i \langle v_2\ e_i\rangle = 0$

So $V = U + U^{\perp}$

If $v \in U \cap U^{\perp}$ then $\langle \underset{U}{v} \in\ \underset{U^{\perp}}{v} \in \rangle = 0$ so $v = 0$

therefore $U \cap U^{\perp} = \{0\}$

Th 3: If $U \subseteq V$ and V is a finite dimensional vector space then
$\dim U^{\perp} = \dim V - \dim U$

Proof since $V = U \oplus U^{\perp}$, $\dim V = \dim(U \oplus U^{\perp}) =$
$= \dim(U) + \dim(U^{\perp})$ so $\dim(U^{\perp}) = \dim V - \dim U$

Def: v_1 in the previous proof
is called the orthogonal projection of v onto U and denoted by $P_U(v)$
thet is given en inner vector space V and a finite dimensional subspace U of V for every $v \in V$ the orthogonal projection of v on U, $P_U(v)$ is the unique vector s.t we can write
$v = P_U(v) + w$ with $P_U(v) \in U$
$w \in U^{\perp}$.

Note:

$$P_U(v) = \langle v\ e_1 \rangle e_1 + \langle v\ e_2 \rangle e_2 + \cdots + \langle v\ e_m \rangle e_m$$

where $e_1\ e_2 \dots e_m$ is en orthonormal basis for U. ($P_U(v)$ does not depend on the choice of the basis)

Th 4: $P_U \quad V \longrightarrow U$

is a linear transformation and $P_U^2 = P_U$

Proof:

choose orthonormal basis for U $B = e_1, \dots e_m$

1) $P_U(v+w) = \sum_{i=1}^{m} \langle v+w \; e_i \rangle e_i = \sum_{i=1}^{m} \langle v \; e_i \rangle +$

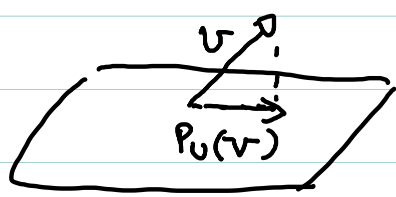
$\sum_{i=1}^{m} \langle w \; e_i \rangle e_i = P_U(v) + P_U(w)$

2) $P_U(kv) = \sum_{i=1}^{m} \langle kv \; e_i \rangle e_i = k \sum_{i=1}^{m} \langle v \; e_i \rangle e_i =$

$= k P(v)$

Since for any $x \in U$ $P_U(x) = x$ because

$x = x + 0$ then $\forall v \in V$ $P_U(P_U(v)) = P_U(v)$



Th 5: Let V be an inner product space, and let U be a finite dimensional subspace of V then $\forall y \in V$ $\forall x \in U$ $\|y - x\| \geq \|y - P_U(y)\|$

Proof: we already proved $y = P_U(y) + z$ with $z \in U^\perp$. For any $x \in U$

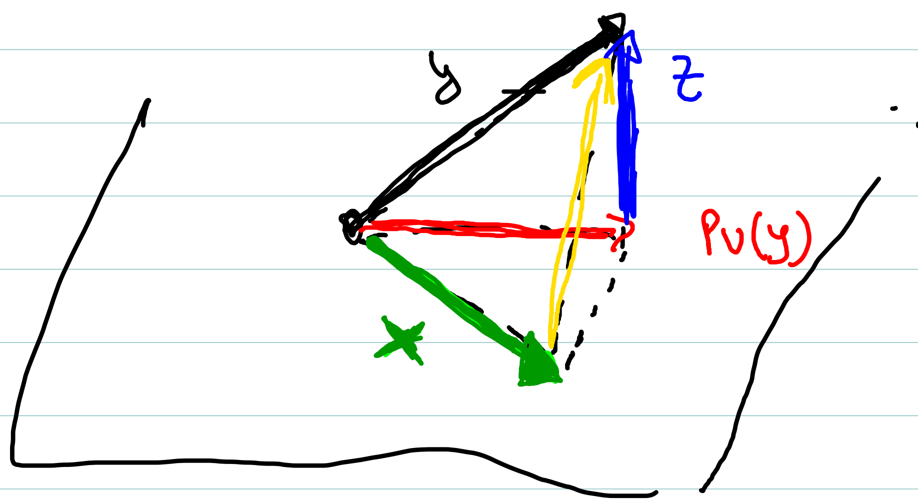
$\|y - P_U(y)\|^2 \leq \|y - P_U(y)\|^2 + \|P_U(y) - x\|^2$ (since $\|P_U(y) - x\|^2 \geq 0$)

$y - P_U(y)$ and $P_U(y) - x$ are perpendicular since $y - P_U(y) = z \in U^\perp$ and $P_U(y) - x \in U$

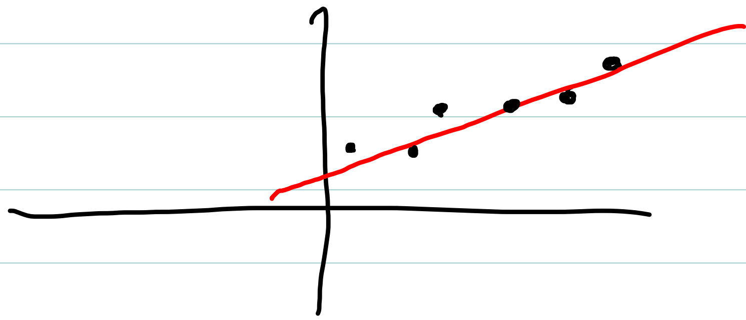
so we can apply the Pythagorean theorem

$\|y - P_U(y)\|^2 + \|P_U(y) - x\|^2 = \|y - P_U(y) + P_U(y) - x\|^2$

Therefore $\|y - P_U(y)\|^2 \leq \|y - x\|^2$



Least square approximation data fitting



Goal: find line that best fits

data points (x_1, y_1) (x_2, y_2) - - - - (x_n, y_n)

That is find line $y = cx + d$ such that

the error. $$E = \sum_{i=1}^{n} (y_i - cx_i - d)^2$$

is as small as possible. If

$$A \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad x = (c, d) \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

then $E = \| y - Ax \|^2$

(c, d) is then the vector
thet minimizes $\| y - Ax \|$

so $A \begin{pmatrix} c \\ d \end{pmatrix}$ is the orthogonal

projection of y on $W = \{ Ax \,/\, x \in R^2 \}$

$$W = \{ Ax \quad x \in R^2 \}$$

Quiz

T F $\text{Proj}_U(y) \neq 0$

F if $y \in U^\perp$ then $y = 0 + y$

T F $\text{Proj}_{U^\perp}(y) \perp \text{Proj}_U(y)$

T since the first vector is in $U^\perp$ and the second in U

$$y = \text{Proj}_{U^\perp}(y) + \text{proj}_U(y)$$