Lesson 23

More properties of norm
6.1

Gram-Schmidt orthogonalization algorithm

6.2

Further properties of norm

Recall $\|v\| = \sqrt{\langle v\, v\rangle}$

Th 1:

d) (Triangle inequality) $\|v+w\| \leq \|v\| + \|w\|$

$\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v+w\rangle + \langle w, v+w\rangle =$

$= \langle v\ v\rangle + \langle v\ w\rangle + \langle w\ v\rangle + \langle w w\rangle$

$= \|v\|^2 + \langle v\ w\rangle + \overline{\langle v\ w\rangle} + \|w\|^2$

$= \|v\|^2 + 2\,\mathrm{Re}\,\langle v, w\rangle + \|w\|^2$

$\leq \|v\|^2 + 2\,|\langle v, w\rangle| + \|w\|^2 \leq$ (since for a complex number z, $z = x+iy$ $\mathrm{Re}\, z = x$

$|z| = \sqrt{x^2+y^2} \geq |x| \geq x = \mathrm{Re}\, z$)

$\leq \|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2$ (by Cauchy Schwartz)

$= (\|v\| + \|w\|)^2$

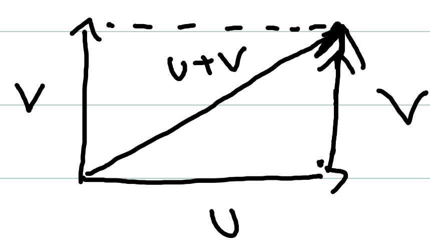
Note If $V = R$ $\langle x\ y\rangle = x \cdot y$ then $\|x\| = |x|$

so the triangle inequality becomes

$|x+y| \leq |x| + |y|$

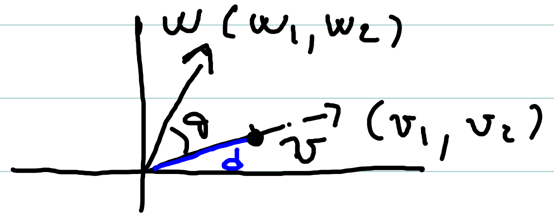
Def Let V be an inner vector space. Then v and w are orthogonal (perpendicular) iff $\langle v, w\rangle = 0$

Pythagorean th: If u and v are orthogonal vectors $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

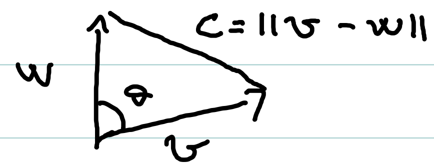


Hw problem

Note: in R^2 $\langle v, w \rangle = \|v\| \, \|w\| \cos\theta$



you can use the law of cosines



$$\|w\|^2 + \|v\|^2 = \|v-w\|^2 + 2\|v\|\,\|w\| \cos\theta$$

$$w_1^2 + w_2^2 + v_1^2 + v_2^2 = (v_1 - w_1)^2 + (v_2 - w_2)^2 + 2\|v\|\|w\|\cos\theta$$

$$2v_1 w_1 + 2v_2 w_2 = 2\|v\|\,\|w\| \cos\theta$$

$$\langle v \quad w \rangle = \|v\|\,\|w\| \cos\theta$$

Def A set S of vectors is orthogonal if for any $v, w \in S$ if $v \neq w$ then $\langle v\ w \rangle = 0$

Def A set S of vectors is orthonormal if for every vector v in S $\|v\| = 1$ and for every pair of vectors v, w in S, with $v \neq w$ $\langle v, w \rangle = 0$

Def An orthonormal basis is a basis that is an orthonormal set

Th 2: An orthonormal set S is linearly independent. (more in general any orthogonal set of non zero vectors)

Proof:

Assume $v_1, v_2, \dots v_n \in S$ and $w = a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$

then $\langle w\ v_i \rangle = a_i \langle v_i\ v_i \rangle = 0$ so $a_i \cdot 1 = 0$

so $a_i = 0$ for $i = 1 \cdots n$.

Th3: If $B = \{b_1, \dots b_n\}$ is an orthogonal set with all $b_i \neq 0$ and $v \in \operatorname{Span}(B)$ then

$$v = \sum_{i=1} \frac{\langle v \; b_i \rangle}{\|b_i\|^2} b_i$$

Proof

$v = a_1 b_1 + \dots + a_n b_n$

$\langle v \; b_i \rangle = a_i \langle b_i \; b_i \rangle$ so $a_i = \langle v \; b_i \rangle / \langle b_i \; b_i \rangle$

Th4: if $B = b_1, b_2 \dots b_n$ is an orthonormal basis for V and $v \in V$ then $v = \sum_{i=1}^{n} \langle v \; b_i \rangle b_i$

Proof: by th3

Def: Let β be an orthonormal subset (possibly infinite) of V and $v \in V$ then the scalars $\langle v \; b \rangle$ for $b \in \beta$ are called Fourier coefficients of v relative to β

Idea: if V is infinite dimensional and has an orthonormal set $\{b_i \mid i \in \mathbb{N}\}$ and $v \in V$, try to write $v = \sum_{i=1}^{\infty} \langle v\, b_i \rangle b_i$.

Not a FINITE linear combination, to make sense of it need a notion of convergence:

$$\lim_{n \to +\infty} \sum_{i=1}^{n} \langle v\, b_i \rangle b_i$$

In a vector space V with a norm $\| \;\|$ we can define $\lim_{n \to +\infty} w_n = w$ iff

$$\forall \varepsilon > 0 \;\; \exists m \;\; \forall n \geq m \quad \| w_n - w \| < \varepsilon$$

Th Gram -Schmidt Every finite dimensional inner vector space has an orthonormal basis.

Proof:

Let $B_1 = w_1, \dots, w_n$ be a basis for V

We will define an orthonormal basis $B_2 = b_1 \dots b_n$ by recursion as follows

$$b_1 = \frac{w_1}{\|w_1\|}$$

$$U_k = w_k - \sum_{j=1}^{k-1} \langle w_k \, b_j \rangle \, b_j \quad , \quad b_k = \frac{U_k}{\|U_k\|}$$

Then for all $k = 1, \dots n$

$C_k = \{ b_1 \dots b_k \}$ is an orthonormal set that spans $V_k = \text{span}\{ w_1 \dots w_k \}$ so $B_2 = C_n$ is a basis for V

by induction on k

$k=1$ obvious

Assume $\{b_1 \cdots b_k\}$ is an orthonormal set that

spans $\text{span}\{w_1 \cdots w_k\}$. First we consider

$$u_{k+1} = w_{k+1} - \sum_{j=1}^{k} \langle w_j\ b_j \rangle b_j$$

then $\langle u_{k+1}\ b_i \rangle = \langle w_{k+1}\ b_i \rangle - \sum_{j=1}^{k} \langle w_{k+1}\ b_j \rangle \langle b_j\ b_i \rangle$

$= \langle w_{k+1}\ b_i \rangle - \langle w_{k+1}\ b_i \rangle = 0$ Therefore

$\langle b_{k+1}\ b_i \rangle = \frac{1}{\|u_{k+1}\|} \langle u_{k+1}\ b_i \rangle = 0$

Therefore $b_1 \cdots b_{k+1}$ is an orthonormal set.
(and so it is independent)

clearly $b_{k+1} \in \text{Span}(w_1 \cdots w_{k+1})$

so $\text{span}(b_1 \cdots b_{k+1}) \subseteq \text{span}(w_1 \cdots w_{k+1})$

and since both spaces have dimension $k+1$

$\text{Span}(b_1 \cdots b_{k+1}) = \text{Span}(w_1 \cdots w_{k+1})$

Ex $W = \text{span}((1\,1\,1)\ (2\,0\,1))$ plane in $\mathbb{R}^3$

$B = (1,1,1)\ (2,0,1)$ not orthonormal

We want to use G.S algorithm to find an orthonormal basis for W

$$\|(1,1,1)\| = \sqrt{\langle (1,1,1)\ (1,1,1)\rangle} = \sqrt{3}$$

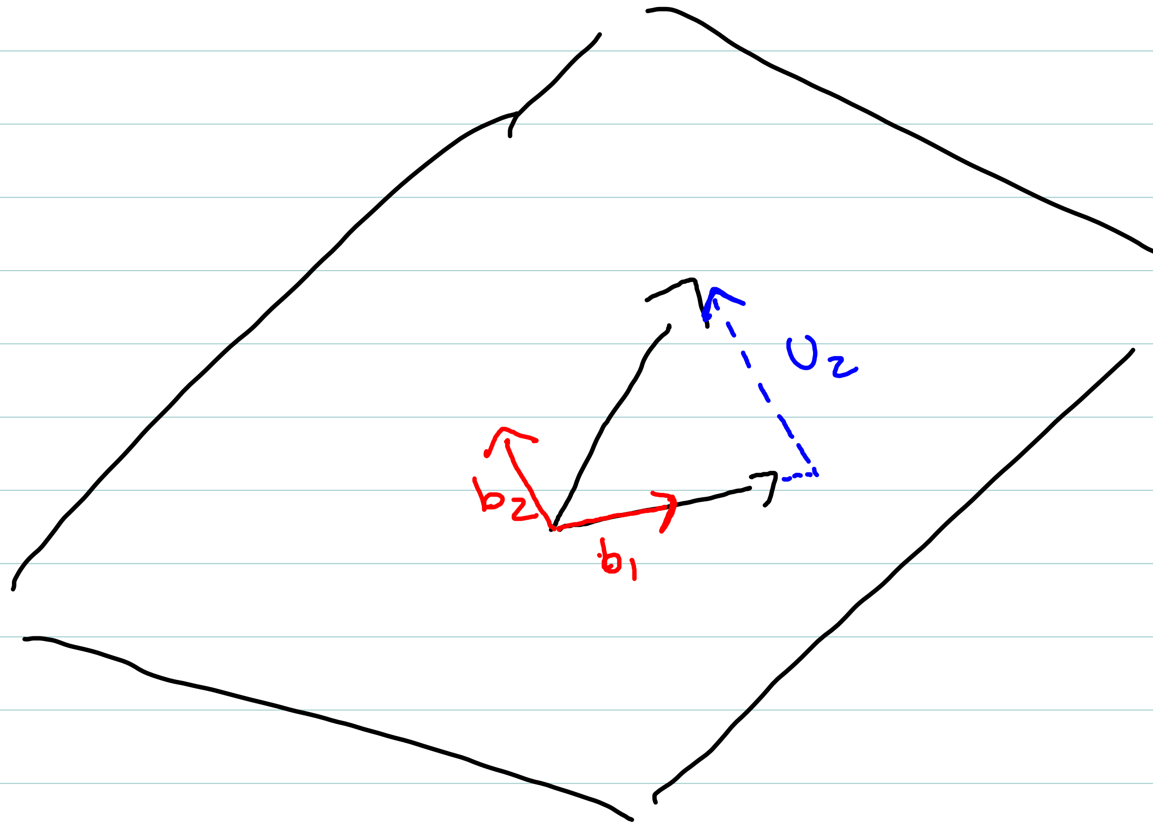
$$b_1 = \frac{1}{\sqrt{3}}(1,1,1)$$

$$U_2 = (2\,0\,1) - \langle (2\,0\,1)\ \tfrac{1}{\sqrt{3}}(1,1,1)\rangle \tfrac{1}{\sqrt{3}}(1,1,1)$$

$$= (2\,0\,1) - \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}(1,1,1) = (1\,{-1}\,0)$$

$$b_2 = \frac{1}{\sqrt{2}}(1,-1,0)$$

$C : \frac{1}{\sqrt{3}}(1,1,1)\ \ \frac{1}{\sqrt{2}}(1,-1,0)$ is an orthonormal basis for W



$$\langle (1\ 1\ 1)_v \quad (2\ 0\ 1)_w \rangle = 3$$

$$3 = \sqrt{3} \cdot \sqrt{5} \cos\theta$$

$$\sqrt{\frac{3}{5}} = \cos\theta \qquad \theta \approx 0.7 \text{ rad}$$

Example consider $P_2(R)$ as a vector space over R with $\langle p, q \rangle = \int_{-1}^{1} p\, q\, dx$

Check $\langle\ \rangle$ is an inner product and find orthonormal basis for $P_2(R)$

1) $\langle p+q, f \rangle = \int_{-1}^{1} (p+q) f\, dx = \int_{-1}^{1} p f\, dx + \int_{-1}^{1} q f\, dx = \langle p\ f \rangle + \langle q\ f \rangle$

2) $\langle cp, f \rangle = \int_{-1}^{1} c p f\, dx = c \int_{-1}^{1} p f\, dx = c \langle p\ f \rangle$

3) $\langle p\ q \rangle = \int_{-1}^{1} pq = \int_{-1}^{1} qp = \langle q\ p \rangle$

4) $\langle p\ p \rangle = \int_{-1}^{1} p^2\, dx > 0$ if $p \neq 0$

if $p \neq 0$ $p(x_0) \neq 0$ for some x_0 so $p^2(x_0) > 0$ in some interval $(x_0 - \varepsilon\ \ x_0 + \varepsilon)$

Therefore $\int_{-1}^{1} p^2\, dx > 0$

Find an orthonormal basis for $P_2(R)$

Start with $\{1, x, x^2\}$ and apply Gram Schmidt:

$$\|1\|^2 = \int_{-1}^{1} 1\cdot 1 \, dx = x\Big|_{-1}^{1} = 2$$

$$b_1 = \frac{1}{\sqrt{2}} \cdot 1$$

$$v_2 = x - \langle x\, b_1\rangle b_1$$

$$\langle x, \frac{1}{\sqrt{2}}\rangle = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}\frac{x^2}{2}\Big|_{-1}^{1} = 0$$

$$\| x\|^2 = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$b_2 = \sqrt{\frac{3}{2}}\, x$$

$$v_3 = x^2 - \langle x^2\, b_1\rangle b_1 - \langle x^2\, b_2\rangle b_2$$

$$\int_{-1}^{1} x^2 \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{\sqrt{2}\,3}$$

$$\int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{2}}\, x\, dx = 0$$

$$v_3 = x^2 - \frac{2}{\sqrt{2}\,3}\cdot\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$$

$$\|v_3\|^2 = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} =$$

$$= \frac{x^5}{5}\Big|_{-1}^{1} - \frac{2}{9}x^3\Big|_{-1}^{1} + \frac{1}{9}x\Big|_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9}$$

$$= \frac{8}{45} \qquad v_3 = 3\sqrt{\frac{5}{8}}\left(x^2 - \frac{1}{3}\right)$$

Orthonormal basis is $\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \ 3\sqrt{\frac{5}{8}}\left(x^2 - \frac{1}{3}\right)$