Lesson 22

Inner product spaces

read 6.1

Complex numbers

$z = x + iy \qquad x = \operatorname{Re} z \qquad y = \operatorname{Im} z$

Usual arithmetic and $i^2 = -1$

Ex $(2+3i) - (1-i) = 1 + 4i$

$(2+3i)(1-i) = 2 - 2i + 3i - 3i^2 = 5 + i$

$z = x+iy \qquad \bar{z} = x - iy$ (conjugate of z)

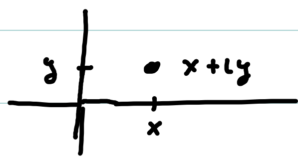
$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2+y^2}$

Note $|z| \in R \qquad |z| \geq 0$

norm of z or modulus of z



if $x \in R \qquad \bar{x} = x$

Inner product spaces

Def Let V be a vector space over F. An inner product on V is a function $f \; V \times V \to F$ s.t the following hold.

a) $f(x+z, y) = f(x,y) + f(z,y)$

b) $f(cx, y) = c\, f(x,y)$

c) $f(x,y) = \overline{f(y,x)}$ (if $F = \mathbb{R}$ this becomes $f(x\,y) = f(y,x)$)

d) $f(x,x) > 0$ if $x \neq 0$ (note that $f(x\,y) = \overline{f(y\,x)}$ implies $f(x\,x) \in \mathbb{R}$ even when $F = \mathbb{C}$)

Ex $\langle\ \rangle \; \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$\langle (x_1, \dots x_n)\ (y_1, \dots y_n) \rangle = x_1 y_1 + \dots + x_n y_n$

Ex $\langle\ \rangle \; \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

$\langle (x_1, \dots x_n)\ (y_1, \dots y_n) \rangle = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$ Note that $\langle v, v \rangle \geq 0$

Ex: $C[0\,1]$ is the space of continuous functions $f\ [0,1] \to \mathbb{R}$

$\langle\ \rangle \; C[0\,1] \times C[0,1] \to \mathbb{R}$

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$$

Th 1: The following are true in an inner product space V

a) $\langle x, y+z\rangle = \langle x\ y\rangle + \langle x, z\rangle$:

$\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \overline{\langle y,x\rangle} + \overline{\langle z, x\rangle}$

$= \langle x, y\rangle + \langle x, z\rangle$

b) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$:

$\langle x, cy\rangle = \overline{\langle cy, x\rangle} = \overline{c\langle y, x\rangle} = \bar{c}\ \overline{\langle y, x\rangle} = \bar{c}\langle x, y\rangle$

c) $\langle x, 0\rangle = \langle 0, x\rangle = 0$:

$\langle 0, x\rangle = \langle 0+0, x\rangle = \langle 0, x\rangle + \langle 0, x\rangle$ therefore $0 = \langle 0, x\rangle$

The proof is similar for $\langle x, 0\rangle = 0$

d) $\langle x, x\rangle = 0$ iff $x = 0$:

by c) and property d) in the definition of $\langle\ \rangle$

e) If $\langle x\ y\rangle = \langle x\ z\rangle$ for all $x \in V$ then $y = z$

Assume $\langle x y\rangle = \langle x z\rangle$ then $\langle x y\rangle - \langle x z\rangle = 0$ so

$\langle x\ y-z\rangle = 0$ for all $x \in V$ take $x = y-z$ then

$\langle y-z, y-z\rangle = 0$ so $y-z = 0$ so $y = z$.

Def Let V be an inner product space. Then

$\|v\| = \sqrt{\langle v\ v \rangle}$

Ex in $\mathbb{R}^2$ $\|(1,2)\| = \sqrt{\langle (1,2)\ (1,2) \rangle} = \sqrt{1^2+2^2} = \sqrt{5}$

Ex in $\mathbb{C}^2$ $\|(i,2)\| = \sqrt{\langle (i,2)\ (i,2) \rangle} = \sqrt{i \cdot \bar{i} + 2 \cdot \bar{2}}$

$= \sqrt{i(-i) + 4} = \sqrt{5}$

Ex $\|1\|$ in $C[0,1]$ $= \sqrt{\int_0^1 1 \cdot 1 \, dx} =$

$= \sqrt{x \big|_0^1} = \sqrt{1} = 1$

Norms allow us to do "calculus" in a vector space

Th2: Let V be an inner product space over F. Then for all $v, w \in V$ and $c \in F$ we have

a) $\|cv\| = |c| \cdot \|v\|$ ($|c|$ = absolute value if $c \in \mathbb{R}$, modulus if $c \in \mathbb{C}$ $|x+iy| = \sqrt{x^2+y^2}$)

$$\sqrt{\langle cv, cv \rangle} = \sqrt{c\bar{c} \langle v\, v \rangle}$$
$$= |c| \cdot \|v\|$$

b) $\|v\| \geq 0$

$$\|v\| = \sqrt{\langle v, v \rangle} \geq 0$$

c) $\|v\| = 0 \iff v = 0$

since $\langle v, v \rangle = 0 \iff v = 0$

d) Cauchy Schwartz : $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$

If $w = 0$ the inequality is true

Assume $w \neq 0$. For any $c \in F$ we have

$$0 \leq \|v - cw\|^2 = \langle v - cw, v - cw \rangle = \langle v\, v \rangle$$
$$- \bar{c} \langle v\, w \rangle - c \langle w\, v \rangle + c\bar{c} \langle w\, w \rangle$$

Take $c = \dfrac{\langle v\, w \rangle}{\langle w\, w \rangle}$ to get

$$0 \leq \langle v\, v \rangle - \frac{\overline{\langle v, w \rangle}}{\langle w\, w \rangle} \cdot \langle v\, w \rangle - \frac{\langle v\, w \rangle}{\langle w\, w \rangle} \overline{\langle v\, w \rangle} + \frac{\langle v\, w \rangle \overline{\langle v\, w \rangle}}{\langle w\, w \rangle^2} \langle w\, w \rangle$$

$$0 \leq \langle v\, v \rangle - \frac{|\langle v\, w \rangle|^2}{\langle w\, w \rangle}$$ so $|\langle v\, w \rangle|^2 \leq \|v\|^2 \|w\|^2$

Example:

Use Cauchy Schwartz inequality to prove

$$(x_1+x_2+\cdots+x_n)^2 \le n(x_1^2+\cdots+x_n^2)$$

for all positive integers n and real numbers x_i

$V = R^n$

n R^n consider the vectors $(1,1,\cdots 1)$ and $(x_1, x_2 \cdots x_n)$ and apply cheuchy Schwartz:

$$|\langle (1,\cdots 1)(x_1, x_2 \ldots x_n)\rangle| \le \|(1,\cdots 1)\| \, \|(x_1 \cdots x_n)\|$$

$x_1 + x_2 + \cdots x_n \le \sqrt{n} \cdot \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$, square both sides

$$(x_1 + \cdots + x_n)^2 \le n \cdot (x_1^2 + \cdots x_n^2)$$