Lesson 21

More about Generalized eigenvectors.

Th 1: Assume $T \in \mathcal{L}(V)$
$\dim V = n$, then
$V = \mathcal{N}(T^n) \oplus \mathcal{R}(T^n)$

Proof: Let $V_1 = \mathcal{N}(T^n)$
$V_2 = \mathcal{R}(T^n)$.
First we prove $V_1 \cap V_2 = \{0\}$
Suppose $v \in V_1 \cap V_2$, then
$T^n(v) = 0$ and $v = T^n(u)$
for some u in V, so $T^{2n}(u) = 0$
so $T^{2n}(u) = 0$ and since

$\mathcal{N}(T^n) = \mathcal{N}(T^{2n})$ then

$u \in \mathcal{N}(T^n)$ which
means $v = T^n(u) = 0$.
Therefore $V_1 + V_2$ is direct and
$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 =$

$\dim V$ (by the rank nullity th
applied to T^n).
Therefore $V = V_1 \oplus V_2$

Th 2: Assume $T \in \mathcal{L}(V)$, $\dim V = n$, $F = \mathbb{C}$, then the generalized eigenvectors of T span V

Proof by induction on n

If $n=1$ then T must have an eigenvalue λ (because $F=\mathbb{C}$) and if v is an eigenvector for λ $V = \operatorname{span}(v)$.

Assume the result is true for all spaces of dimension less than n, and let $\dim V = n$. V has at least one eigenvalue λ (because $F=\mathbb{C}$) and $V = N(T-\lambda I)^n \oplus R(T-\lambda I)^n$ by th 1 applied to $T-\lambda I$

$N(T-\lambda I)^n \neq \{0\}$ since T has eigenvectors so $\dim R(T-\lambda I)^n < n$

$R(T-\lambda I)^n$ is invariant under T since if $v \in R(T-\lambda I)^n$ that is $v = (T-\lambda I)^n w$ for some w in V then $Tv = T(T-\lambda I)^n w$ $= (T-\lambda I)^n (Tw)$ is in $R(T-\lambda I)^n$

Our goal is to show $R(T-\lambda I)^n$ is spanned by generalized eigenvectors of T.

Let $V_2 = R(T - \lambda I)^n$

We can consider $T_{/V_2}\ V_2 \to V_2$ and apply our induction hypothesis

$V_2 = \text{span}(\text{generalized eigenvectors of } T_{/V_2})$

Suppose λ_i is an eigenvalue for $T_{/V_2}$ and $w \in V_2$ a generalized eigenvector for λ_i: then

$T_{/V_2}\, v = \lambda_i v$ for some v in V_2

but then $T v = \lambda_i v$ so λ_i is an eigenvalue for T and

$(T_{/V_2} - \lambda_i I)^k w = 0$ for some k

and $(T_{/V_2} - \lambda_i I)^k w = (T - \lambda_i)^k w$

so w is a generalized eigenvector of T.

so $V_2 = R(T-\lambda I)^n$ is spanned by generalized eigenvectors of T

$V_1 = N(T-\lambda I)^n = K_\lambda$ is spanned by generalized eigenvectors of T

Therefore $V = N(T-\lambda I)^n \oplus R(T-\lambda I)^n$ is spanned by generalized eigenvectors of T.

Th3: Let $T \in \mathcal{L}(V)$, $\dim V = n$
$F = \mathbb{C}$ and let $\lambda_1, \lambda_2, \dots \lambda_m$
be the distinct eigenvalues for T
Then :

(1) $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_m}$

(2) K_{λ_i} is invariant for T
for all $i = 1, \dots m$

(3) There is a basis B of V
made of generalized eigenvectors of
T such that T_B^B is a block matrix

$$T_B^B = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}$$

Each matrix A_i is a $n_i \times n_i$ matrix, where
n_i is the dimension of K_{λ_i} $\quad A_i = \begin{bmatrix} \lambda_i & * & * \\ 0 & \ddots & * \\ \vdots & \cdots\ 0 & \lambda_i \end{bmatrix}$

In particular T_B^B is upper triangular

Let's choose bases B_i for K_{λ_i} in the following way:

remember that

$$E_{\lambda_i} = N(T-\lambda_i I) \subset N(T-\lambda_i I)^2 \subset N(T-\lambda_i I)^3 \cdots \subset N(T-\lambda_i I)^{h_i} = N(T-\lambda_i I)^{h_i+1} = \cdots = \underbrace{N(T-\lambda_i I)^n}_{K_{\lambda_i}}$$

$v_1 \cdots v_k$ basis for $N(T-\lambda_i I)$

$v_1 \cdots v_k \; v_{k+1} \cdots v_{k+j_1}$ basis for $N(T-\lambda_i I)^2$

$v_1 \cdots v_k \; v_{k+1} \cdots v_{k+j_1} \; v_{k+j_1+1} \cdots v_{k+j_1+j_2}$ is a basis for $N(T-\lambda_i I)^3$

$$\underbrace{v_1 \cdots v_k}\; \underbrace{v_{k+1} \cdots v_{k+j_1}}\; \underbrace{v_{k+j_1+1} \cdots v_{k+j_1+j_2}} + \cdots\cdots \; v_{n_i}$$

is a basis for K_{λ_i}

$B = \bigcup B_i$

What does T_B^B look like?

Recall each K_{λ_i} is T invariant

$$T_B^B = \begin{array}{cc} & \begin{array}{ccccc} \overbrace{T(b_1)\;\cdots}^{B_1} & \overbrace{T(b_j)}^{B_2} & \cdots & \cdots & T(b_n) \end{array} \\ \begin{array}{c} B_1\{\, b_1 \\ \vdots \\ B_2\{\, \vdots \\ \vdots \\ B_m\{\, b_n \end{array} & \left[\begin{array}{cccc} A_1 & 0 & & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{array}\right] \\ & \begin{array}{cccc} \underbrace{\qquad}_{B_1} & \underbrace{\qquad}_{B_2} & \cdots & \underbrace{\qquad}_{B_m} \end{array} \end{array}$$

What does A_l look like?

$$\begin{array}{c} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ \\ \vdots \\ v_{n_l} \end{array}
\begin{array}{c} T(v_1) \;\cdots\; T(v_k) \;\; T(v_{k+1}) \;\; T(v_{k+2}) \;\cdots\; T(v_{n_l}) \\
\begin{bmatrix}
\lambda_l & 0 & \cdots & 0 & \times & \times & & \times \\
0 & \lambda_l & & \vdots & \times & \times & & \times \\
\vdots & \vdots & \ddots & \lambda_l & \times & \times & & \times \\
\vdots & \vdots & & 0 & \lambda_l & 0 & & \times \\
\vdots & \vdots & & \vdots & 0 & \lambda_l & & \times \\
\vdots & \vdots & & \vdots & \vdots & 0 & \ddots & \times \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \lambda_l
\end{bmatrix}\end{array}$$

$T(v_1) = \lambda_l v_1 \cdots \; T(v_k) = \lambda_l v_k$

$$T(v_{k+1}) = \underbrace{(T - \lambda_l I)\, v_{k+1}}_{\in N(T-\lambda_l I)} + \lambda_l v_{k+1}$$

$$T(v_{k+j_1+1}) = \underbrace{(T - \lambda_l I)\, v_{k+j_1+1}}_{\in N(T-\lambda_l I)^2} + \lambda_l I\, v_{k+j_1+1}$$

$$T(v_{n_l}) = \underbrace{(T - \lambda_l I)\, v_{n_l}}_{\in N(T-\lambda_l I)^2} + \lambda_l I\, v_{n_l}$$

n_i is the dim of K_{λ_i}
n_i is the algebraic multeplicity of K_{λ_i}

$n_i \geq h_i$ so $K_{\lambda_i} = N(T - \lambda I)^{h_i}$

Th 4: : Let $T \in \mathcal{L}(V)$ $\dim V = n$

λ an eigenvalue of T with algebraic multeplicity m. Then $\dim(K_\lambda) = m$

and $K_\lambda = N(T - \lambda I)^m$

Proof Consider the proof of the previous th where $\dim(K_{\lambda_i}) = n_i$ and all the vectors in the chosen basis of K_{λ_i} are in $N(T-\lambda_i)^{h_i}$. $n_i \geq h_i$ so $K_{\lambda_i} = N(T - \lambda_i)^{n_i}$. T_B^B in the proof of the previous th. has n_i λ_i on the diagonal and it has characteristic polynomial
$(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdots (x-\lambda_m)^{n_m}$

so $n_i =$ algebraic multeplicity of λ_i

Jordan canonical form

For a better chosen $\mathcal{B}$

$T_{\mathcal{B}}^{\mathcal{B}}$ has the form

$$\begin{bmatrix} \lambda_1 & * & & & & & & 0 \\ & \ddots & * & & & & & \\ & & \lambda_1 & * & & & & \\ & & & \lambda_2 & * & & & \\ & & & & \ddots & * & & \\ & 0 & & & & \ddots & * & \\ & & & & & & \ddots & * \\ & & & & & & & \lambda_m \end{bmatrix}$$

$*$ can be 0 or 1