Lesson 20

Linear operators (matrices) over $\mathbb{C}$ - Generalized eigenspaces

see ch 7

Recap : given an $n \times n$ matrix A with entries in F (so here $V = F^n$) or an operator $T : V \to V$ with $\dim(V) = n$

If $\lambda_1, \lambda_2, \dots \lambda_m$ are the distinct eigenvalues of A/T, A/T is diagonalizable iff

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m}$$

Problem 1

$F = \mathbb{R}$

A $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ has no real eigenvalues

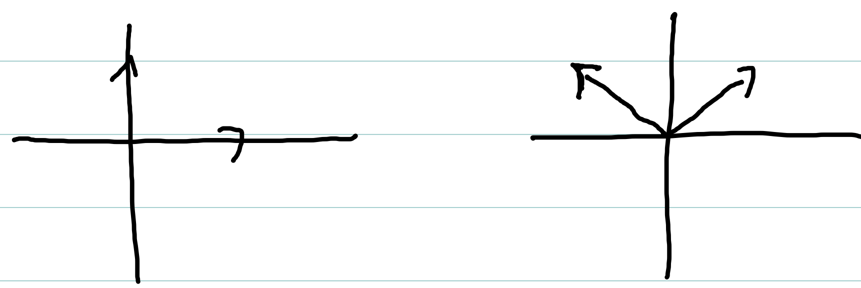
Characteristic polynomial is

$p(x) = (1-x)^2 + 1$

Geometric interpretation

$$A = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}$$

rotation

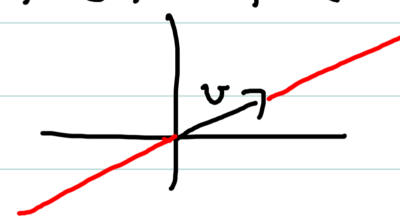


No fixed lines = no eigenvalues

since if $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(v) = Av$

has an eigenvalue λ with eigenvector v then the line $(x, y) = \lambda v$



is fixed by T.

$$p(x) = (1-x)^2 + 1$$

In $\mathbb{C}$ $\quad p(x) = (x-(1+i))(x-(1-i))$

Th (Fundamental th of Algebra)

every polynomial $p(x)$ splits over $\mathbb{C}$

Corollary 1 : Every $T \in \mathcal{L}(V)$ where $\dim V = n$ vector space over $\mathbb{C}$ has at least one eigenvalue

Proof Let $p(x)$ be the characteristic polynomial of T then

$$p(x) = c\,(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

so T has at least one eigenvalue (it could be only one if $\lambda_1 = \lambda_2 \cdots = \lambda_n$)

Example of $T \in \mathcal{L}(V)$, V a complex vector space. T has no eigenvalues. V is not finite dimensional.

$$T \quad \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$$

$$T(x_1\ x_2 \cdots x_n \cdots) = (0\ x_1\ x_2 \cdots)$$

Let: $V = \{x_n\} = x_1\ x_2 \cdots x_n \cdots$

Can $T(\{x_n\}) = \lambda\{x_n\}$ for some $\lambda \in \mathbb{C}$?

If $T(\{x_n\}) = \lambda\{x_n\}$ then

$(0\ x_1\ x_2 \cdots) = (\lambda x_1\ \lambda x_2 \cdots)$ therefore

if $\lambda \neq 0$ we must have $x_i = 0 \ \forall i$

so $\{x_n\} = 0$, not an eigenvector

if $\lambda = 0$, still $\{x_n\} = 0$

so T has no eigenvalues

Problem 2

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

char poly : $p(x) = x^2$

eigenvalues $\lambda = 0$, $\lambda = 0$

$E_0 = \{ (x, 0) \mid x \in \mathbb{C} \}$ which has dimension 1

A is not diagonalizable.

$$\mathbb{C}^2 \neq E_0$$

0 has geometric multiplicity 1 and algebraic multiplicity 2

Therefore it is possible that

$$E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m} \lneq V$$

If we want to write V as sum of invariant subspaces of T, we need more vectors.

Def : Let $T \in \mathcal{L}(V)$ and λ_i be an eigenvalue for T, w is a generalized eigenvector for λ_i if $w \neq 0$ and $w \in N(T-\lambda_i I)^k$ for some k in $\mathbb{N}$

Note : if $w \in N(T-\lambda_i I)^k$ and k is the smallest positive integer such that this is true, then $(T-\lambda_i I)^{k-1} w$ is an eigenvector for λ_i

Th 1:

1) $N(T-\lambda I)^k \subseteq N(T-\lambda I)^{k+1}$ for all $k \geq 1$

if $(T-\lambda I)^k v = 0$ then

$(T-\lambda I)^{k+1} v = (T-\lambda I)(T-\lambda I)^k v = 0$

2) if $N(T-\lambda I)^k = N(T-\lambda I)^{k+1}$

then $N(T-\lambda I)^k = N(T-\lambda I)^{k+m}$

for all $m \geq 1$

By induction on m

Base case: clear

Induction step: assume $N(T-\lambda I)^k = N(T-\lambda I)^{k+m}$

$N(T-\lambda I)^k \subseteq N(T-\lambda I)^{k+m}$ so we

only need to show $N(T-\lambda I)^{k+m+1} \subseteq N(T-\lambda I)^k$

Assume $v \in N(T-\lambda I)^{k+m+1}$

then $(T-\lambda I)^{k+m+1} v = 0$ so

$(T-\lambda I)^{k+m}(T-\lambda I) v = 0$

so $(T-\lambda I) v \in N(T-\lambda I)^k$

So $v \in N(T-\lambda I)^{k+1} = N(T-\lambda I)^k$

Note: If $\dim V = n$ and $\dim E_\lambda = p$

$$E_\lambda = \underset{\dim p}{N(T-\lambda I)} \subset \underset{\text{dim at least } p+1}{N(T-\lambda I)^2} \subset \cdots \subset \underset{\text{dim at least } p+k-1}{N(T-\lambda I)^k}$$

So we may have proper inclusion but eventually we need to stop because V has finite dimension

Th2: If $T \in \mathcal{L}(V)$, $\dim V = n$, λ is an eigenvalue for V, then any generalized eigenvector for λ is in $N(T-\lambda I)^n$

Proof: by the discussion above

Note: the discussion above did not use the fact λ is an eigenvalue for T, so it is true that if $T \in \mathcal{L}(V)$ $\dim V = n$

$$N(T^n) = N(T^{n+k}) \quad \text{for all } k \geq 0$$

Def: If $T \in \mathcal{L}(V)$, $\dim V = n$, λ is an eigenvalue for V, then we call $N(T-\lambda I)^n$ the generalized eigenspace of λ and denote it K_λ

Th 3 Assume $T \in \mathcal{L}(V)$, $\dim(V) = n$ and λ is an eingenvalue of V

Then K_λ is T invariant

Proof: Assume $v \in K_\lambda$ then

$$T(T-\lambda I)^n v = (T-\lambda I)^n T v = 0$$

so $Tv \in K_\lambda$

Th4: Suppose $T \in \mathcal{L}(V)$, $\dim V = n$, $\lambda_1, \lambda_2, \cdots \lambda_m$ are distinct eigenvalues for T and $w_1, w_2, .. w_m$ are corresponding generalized eigenvectors. Then $w_1, w_2, \cdots w_m$ are linearly independent.

Proof: assume $a_1 w_1 + a_2 w_2 + \cdots + a_m w_m = 0$ (*)
Let k be the largest integer s.t $(T - \lambda_1 I)^k w_1 \neq 0$, $k \geq 0$, Then $(T-\lambda_1 I)^k w_1 = w$ is an eigenvector for λ_1 $\therefore Tw = \lambda_1 w$ and if $i \neq 1$
$(T - \lambda_i I) w = (\lambda_1 - \lambda_i) w$

Apply $(T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n$ to both sides of (*). Note that $(T-\lambda_1 I)^k$, $(T-\lambda_2 I)^n$, $\ldots (T-\lambda_m I)^n$ commute so we get

$$a_1 (\lambda_1 - \lambda_2)^n (\lambda_1 - \lambda_3)^n \cdots (\lambda_1 - \lambda_m)^n w = 0$$

so $a_1 = 0$

In a similar way we get $a_2 = \cdots a_m = 0$

Th5: Suppose $T \in \mathcal{L}(V)$, $\dim V = n$, $\lambda_1, \lambda_2, \cdots \lambda_m$ are distinct eigenvalues for T then The sum

$$K_{\lambda_1} + K_{\lambda_2} + \cdots + K_{\lambda_m}$$ is direct

Proof: suppose $0 = v_1 + \cdots + v_m$ with $v_i \in K_{\lambda_i}$. If any of these vectors were non zero they would be linearly independent and add up to 0, impossible