Lesson 2

Subspaces

1.3

Def : A subset S of a vector space V is a subspace of V (sometimes I will write $S \leq V$) if

1) $0_V \in S$
2) $u, v \in S \Rightarrow u+v \in S$ for all u, v in S
3) $u \in S$, $\lambda \in F \Rightarrow \lambda u \in S$ for all $\lambda \in F$
   $u \in S$

So a subspace S of V is itself a vector space with the same addition and scaler multiplication as V.

Condition 1) assures a subspace is not empty.

Examples of subspaces:

Ex 1: $V = R^2$

$S = \{(x,y) \mid x+y=0\}$

Check

1) $(0,0) \in S$

2) If $(x,y) \in S$ and $(z,t) \in S$ then $x+y=0$, $z+t=0$

therefore $(x+z)+(y+t)=0$ so $(x+z, y+t) \in S$

3) If $(x,y) \in S$ then $x+y=0$ so $\alpha(x+y)=0$

for any $\alpha \in R$ so $\alpha x + \alpha y = 0$ so

$(\alpha x, \alpha y) \in S$

Ex 2: $T = \{(x,y) \mid x+y=1\}$

$(0,0) \notin T$ so not a subspace.

Describe all subspaces of R^2

1) $\{0 = (0,0)\}$ 2) R^2 3) Any line through origin

Ex: $S = \{\{x_n\} \mid \lim_{n\to+\infty} x_n = 0\} \leq R^\infty$

1) The 0 sequence i.e $x_n = 0$ for all n is in S

2) If $\{x_n\}, \{y_n\} \in S$ we need to show $\{x_n\} + \{y_n\} = \{x_n + y_n\} \in S$:

If $\lim_{n\to+\infty} x_n = 0$ and $\lim_{n\to+\infty} y_n = 0$

then $\lim_{n\to+\infty} (x_n + y_n) = 0$.

so $\{x_n\} + \{y_n\} = \{x_n + y_n\} \in S$

3) If $\{x_n\} \in S$ then $\lim_{n\to+\infty} x_n = 0$ so $\forall \alpha \in R$

$\lim_{n\to+\infty} \alpha x_n = 0$ so $\alpha\{x_n\} = \{\alpha x_n\} \in S$

Ex What about $S = \{\{x_n\} \mid x_n \in R, \lim_{n\to+\infty} x_n = 1\}$?

$0 \notin S$ so not a vector space

Ex $S = \{(x,y) \mid x \in \mathbb{Z},\ y \in \mathbb{Z}\} \subseteq \mathbb{R}^2$

$(0,0) \in S$

if (x,y) and $(z,t) \in S$ then $(x+z,\ y+t) \in S$

But $(1,1) \in S$ $\sqrt{2}(1,1) \notin S$

So S is NOT a subspace of $\mathbb{R}^2$

Ex $S = \{(x\ 0) \mid x \in \mathbb{R}\} \cup \{(0\ y)\ y \in \mathbb{R}\} \subseteq \mathbb{R}^2$

S contains $(0,0)$ and if $(x\ y) \in S$ then $\alpha(x\ y) \in S$ $\forall \alpha \in \mathbb{R}$

But $(1,0)$, $(0,1) \in S$ and $(1,0)+(0,1) \notin S$

So S is NOT a subspace of $\mathbb{R}^3$

Set operations and subspaces.

Th Let $S \leq V$, $T \leq V$ then
$S \cap T \leq V$

Proof :

1) $0_V \in S$, $0_V \in T$ therefore
$0_V \in S \cap T$

2) 3) If $x \in S \cap T$ and $y \in S \cap T$
Then $x \in S$, $x \in T$, $y \in S$, $y \in T$
Therefore $x+y$ and λx belong
to both S and T so
$x+y \in S \cap T$ and $\lambda x \in S \cap T$

Example : $S = \{(x\ y\ z) \mid 3x+2y-z=0\} \leq \mathbb{R}^3$
$T = \{(x\ y\ z) \mid x+y+z=0\} \leq \mathbb{R}^3$

$S \cap T$ is a line through origin

Let $V = \mathbb{R}^2$

$S = \{(x, 0) \mid x \in \mathbb{R}\}$

$T = \{(0, y) \mid y \in \mathbb{R}\}$

is $S \cup T$ a subspace of $\mathbb{R}^2$?

No: $(1,0) \in S \cup T$, $(0,1) \in S \cup T$

but $(1,0) + (0,1) = (1,1) \notin S \cup T$