Lesson 19

5.2 the cherecteristic polynomiel

Def If A is an $n \times n$ matrix
$\det(A - xI) = p(x)$ is called
the characteristic polynomial of A

Note the eigenvalues of A
are the solutions of $p(x) = 0$

Def If $T \in \mathcal{L}(V)$, V is finite
dimensional and B is a basis
for V then the characteristic
polynomial of T is the characteristic
polynomial of T^B_B

Note: if C is a different basis for V then $T_C^C = (I_C^B)^{-1} T_B^B I_C^B$ and

$$\det(T_C^C - \lambda I) = \det((I_C^B)^{-1} T_B^B I_C^B - \lambda I)$$

$$= \det((I_C^B)^{-1} T_B^B I_C^B - \lambda (I_C^B)^{-1} I_C^B) =$$

$$= \det((I_C^B)^{-1} (T_B^B - \lambda I) I_C^B)$$

$$= \det((I_C^B)^{-1}) \det(T_B^B - \lambda I) \det(I_C^B)$$

$$= \det(T_B^B - \lambda I)$$

so the definition above does not depend on the basis we choose

Th 1 If A is a $n \times n$ metrix with entries in $P_1(\mathbb{R})$ det A is a polynomiel in $P_n(\mathbb{R})$

Proof: by induction on n

Base cese: if $n=1$ $A = (a\lambda + b)$
$\det(A) = a\lambda + b$

Induction step: assume the theorem is true for n and A is a $(n+1) \times (n+1)$ metrix

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} \underbrace{a_{1j}}_{\in P_1(\mathbb{R})} \underbrace{\det(A_{1j})}_{\in P_n(\mathbb{R})}$$

is sum of polynomiels in $P_{n+1}(\mathbb{R})$ so it is a polynomiel in $P_{n+1}(\mathbb{R})$

Th 2: If $A=(a_{ij})$ is $n \times n$ $\det(A-\lambda I)$ is a polynomial in λ of degree n

$p(\lambda)=(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)+g(\lambda)$

where $g(\lambda) \in P_{n-2}(R)$

proof by induction on n

Base case: if $n=2$ $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}$

Induction step: assume the theorem is true for n and A is $(n+1) \times (n+1)$ λ

$$B = A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n+1} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ a_{n+1 1} & \cdots & & a_{n+1\, n+1} - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (a_{11} - \lambda)\det(B_{11}) +$$

$$+ \sum_{j=2}^{n+1} (-1)^{1+j} a_{1j} \det(B_{1j})$$

$B_{11} = (A_{11} - \lambda I)$ by induction assumption $\det(B_{11}) =$

$$(a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{nn} - \lambda) + \underbrace{h(\lambda)}_{\in P_{n-2}(R)}$$

so $(a_{11} - \lambda)\det(B_{11}) = (a_{11} - \lambda) \cdots (a_{nn} - \lambda) + \underbrace{(a_{11} - \lambda) h(\lambda)}_{\in P_{n-2}(R)}$

What about $(-1)^{1+j} a_{1j} \det(B_{1j})$ for $j > 1$?

$$B_{1j} = C = \begin{pmatrix} a_{22} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n+1\,1} & a_{n+1\,2} & & a_{n+1\,n+1}-\lambda \end{pmatrix} \in P_{n-2}(\mathbb{R})$$

if we expend along j row we get

$$\sum_{k \neq j} (-1)^{1+j} a_{1j} \, (-1) \, (-1)^{j+k} a_{kj} \underbrace{\det(C_{jk})}_{\in P_{n-1}(\mathbb{R})}$$

Definition: a polynomial $p(x)$ in $P(F)$ splits over F if we can write $p(x) = c\,(x-a_1)(x-a_2)\cdots(x-a_n)$

$a_1, a_2, \dots a_n$ are called the roots of p, they are not necessarily all distinct.

Th 3: If $T \in \mathcal{L}(V)$, where $\dim V < +\infty$ is diagonalizable, then $p(x)$, the characteristic polynomial of T splits

Proof: Let B be a basis s.t T_B^B is diagonal

$$T_B^B = \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{bmatrix}$$ then

$$p(x) = (k_1 - x)(k_2 - x)\cdots(k_n - x)$$

Def : Let λ be an eigenvalue for an operator or a matrix then $\dim(E_\lambda)$ is called the geometrix multeplicity of λ. If $p(x)$ is the

Characteristic polynomial of the operator / matrix the largest positive k s.t $(x-\lambda)^k$ is a factor of $p(x)$ is called the algebraic multeplicity of λ

Th 4: Let $T \in \mathcal{L}(V)$, $\dim V < +\infty$ λ an eigenvalue for T. Then the geometric multeplicity of λ is less then or equal to the algebraic multeplicity of λ

Proof: Let p be the geometric multeplicity of λ and m the algebraic multeplicity.

choose a basis $v_1 \cdots v_p$ for E_λ
and extend it to a basis
$B = v_1 \cdots v_p \cdots v_n$ for V

$$T_B^B = \begin{bmatrix} \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} & B \\ 0 & C \end{bmatrix}$$

$$T_B^B - xI = \begin{bmatrix} \begin{bmatrix} \lambda-x & & 0 \\ & \ddots & \\ 0 & & \lambda-x \end{bmatrix} & B \\ 0 & C-xI \end{bmatrix}$$

$$\det(T_B^B - xI) = (\lambda-x)^p \cdot \det(C-xI)$$

$$= (\lambda-x)^p \, g(x)$$

so $(\lambda-x)^p$ is a factor of the characteristic polynomial of T_B^B so the algebraic multiplicity is at least p
($g(x)$ could have more factors)

Th 5 : Let $T \in \mathcal{L}(V)$ $\dim V = n$
and $F = C$

Let $\lambda_1 \dots \lambda_k$ be the distinct eigenvalues of T. Then T is diagonalizable iff for all i the geometric multeplicity of λ_i is equal to the algebraic multeplicity.

Proof: assume T is diagonalizeble and $\dim(E_{\lambda_i}) = p_i$
$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ therefore
$n = p_1 + p_2 + \cdots + p_k$
so the charecteristic polynomial must be $\pm (x-\lambda_1)^{p_1} (x-\lambda_2)^{p_2} \dots (x-\lambda_k)^{p_k}$

assume that for all i
p_i = algebraic multeplicity of λ_i
then $p(x)$ splits in the product of linear factors and we have p_1 factors equal to $(x-\lambda_1)$, p_2 factors equal to $(x-\lambda_2)\dots$ therefore
$p_1 + p_2 + \cdots + p_k = n$ so

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots + E_{\lambda_k}.$$

Example $T \quad P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$$T(p) = p'(x)(x+1)$$

Is it linear?

$$T(p+q) = (p+q)'(x)(x+1) = p'(x)(x+1) + q'(x)(x+1)$$
$$= T(p) + T(q)$$
$$T(kp) = (kp)'(x)(x+1) = k\, p'(x)(x+1) = kT(p)$$ yes

Is T diagonalizable? Choose $B = 1, x, x^2$

$$T(1) = 0 \qquad T_B^B \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$
$$T(x) = x+1$$
$$T(x^2) = 2x(x+1) = 2x^2 + 2x$$

Eigenvalues $0, 1, 2$

Then it must be diagonalizable since

$$3 \le \dim E_0 + \dim E_1 + \dim E_2 = \dim(E_0 \oplus E_1 \oplus E_2) \le \dim P_2(\mathbb{R}) = 3$$

NOTE If $T \in \mathcal{L}(V)$ $\dim V = n$ and T has n distinct eigenvalues, then T is diagonalizable.

Let's find C st T_C^C is diagonal

$E_0 = N(T - 0I)$, to finde

basis for E_0, first

consider $N(T_B^B - 0I)$

that is solve

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

reduce to echelon form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2
nullity = 1

one non zero solution is $(1,0,0)$

so $N(T_B^B - 0I) = \text{span}(1,0,0)$

$E_0 = \text{span}(1 \cdot 1 + 0 \cdot x + 0 \cdot x^2)$

$E_1 = N(T - I)$, to finde

basis for E_1, first

consider $N(T_B^B - I)$

that is solve

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

reduce to echelon form

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2
nullity = 1

one non zero solution is $(1, 1, 0)$

so $N(T_B^B - I) = \text{span}(1, 1, 0)$

$E_1 = \text{span}(1 \cdot 1 + 1 \cdot x + 0 \cdot x^2)$

$E_2 = N(T - 2I)$, to finde basis for E_2, first consider $N(T^B_B - 2I)$

that is solve

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

rank = 2
nullity = 1

one non zero solution is $(1, 2, 1)$

so $N(T^B_B - 2I) = \text{span}(1, 2, 1)$

$$E_2 = \text{span}(1 \cdot 1 + 2 \cdot x + 1 \cdot x^2)$$

$$C = \{1, 1+x, 1+2x+x^2\}$$