Lesson 18

Eigenspaces E_λ

diagonalizability

5.2

308	340
A is $n \times n$	$T \quad V \to V$

λ is eigenvalue
$v \neq 0$ eigenvector

$Av = \lambda v$	$Tv = \lambda v$
$(A - \lambda I)v = 0$	$(T - \lambda I)v = 0$
$v \in N(A - \lambda I)$	$v \in N(T - \lambda I)$
$A - \lambda I$ not invertible	$T - \lambda I$ not invertible
$\det(A - \lambda I) = 0$	

Eigenvalues and eigenvectors of A are those of $L_A \; F^n \to F^n$ $L_A v = A \cdot v$	If $\dim V < \infty$ $B = \{v_1, \dots v_n\}$ is a basis for V eigenvalues of T are those of T_B^B

and if $(x_1, \dots x_n)$ is an eigenvector for T_B^B then $x_1 v_1 + \cdots + x_n v_n$ is an eigenvector for T

Def Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue for T then

$$N(T-\lambda I) = \{v \mid Tv = \lambda v\}$$

is called the eigenspace of λ and it is denoted by $E(\lambda, T)$ or $E_\lambda(T)$ or E_λ

It is the set of all eigenvectors of T with eigenvalue λ, plus the 0 vector

Th 1: If $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T $E_\lambda \leq V$ and E_λ is T invariant.

Proof: $E_\lambda = N(T-\lambda I)$ and we know the null space of a linear transformation is a subspace of the domain. Since if $v \in E_\lambda$ $T(v) = \lambda v \in E_\lambda$ E_λ is T invariant

Th 2 : Suppose that $\lambda_1 \cdots \lambda_m$ are distinct eigenvalues of T and $v_1 \cdots v_m$ are corresponding eigenvectors. Then $v_1 \cdots v_m$ are linearly independent.

Proof : by induction on m

If $m = 1$ $\quad v_1 \neq 0$ so it is linearly independent

Assume statement is true for $m = k-1$, and that $v_1 \cdots v_k$ are eigenvectors for distinct eigenvalues.

If $(*)$ $\quad a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$ $\quad$ Then

$$(T - \lambda_k I)(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) = 0 \quad \text{so}$$

$$(\lambda_k a_1 - \lambda_1 a_1) v_1 + (\lambda_k a_2 - \lambda_2 a_2 v_2) + \cdots (\lambda_k a_{k-1} - \lambda_{k-1} a_{k-1}) v_{k-1} = 0$$

so we must have $\quad a_1(\lambda_k - \lambda_1) = a_2(\lambda_k - \lambda_2) = \cdots = a_{k-1}(\lambda_k - \lambda_{k-1}) = 0$

since $\lambda_k \neq \lambda_i$ for $1 \le i \le k-1$ we must have $a_1 = a_2 \cdots = a_{k-1} = 0$ and since $v_k \neq 0$ looking at $(*)$ we must have $a_k = 0$ so $v_1\ v_2 \cdots v_k$ are linearly independent.

Th 3 If $T \; V \to V$ is a linear transformation and $\lambda_1 \dots \lambda_n$ are distinct eigenvalues for T the sum $E_{\lambda_1} + \dots + E_{\lambda_n}$ is direct

Proof We just need to show that if $v_1 + v_2 + \dots + v_m = 0$ where $v_i \in E_{\lambda_i}$ then all v_i are 0. This is true since eigenvectors corresponding to different eigenvalues are independent

Th4: Let V be finite dimensional and $T \in \mathcal{L}(V)$; let $\lambda_1, \lambda_2 \cdots \lambda_m$ be the distinct eigenvalues of T. Then the following are equivalent

1) T is diagonalizable
2) V has a basis consisting of eigenvectors of T
3) $V = E_{\lambda_1} \oplus \cdots\cdots \oplus E_{\lambda_m}$
4) $\dim V = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m}$

Proof: 1) $\Leftrightarrow$ 2)

T is diagonalizable iff there is a basis B s.t $T_B^B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

which means that for any b_i in B
$T(b_i) = \lambda_i b_i$, which means B is a basis of eigen vectors of T

2) $\Rightarrow$ 3) If $B = v_1 \cdots v_n$ is a basis of eigenvectors any $v \in V$ can be written as a sum of vectors in E_{λ_i} so $V = E_{\lambda_1} + E_{\lambda_2} + \cdots E_{\lambda_m}$ and we already proved the sum is direct so $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots + E_{\lambda_m}$

3) => 4) Proved in lesson 6

4) => 2) Suppose $\dim V = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m} = n$

Then $E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_n}$ is a direct sum (by Lesson 6) and $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m}$ because they have the same dimension

Let B_i be a basis for E_{λ_i}
then $B = \cup B_i$ has n eigenvectors in it and it spans V since $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$
so it is a basis for V

Def $C^\infty(R)$: functions f $R \to R$

s.t $f^{(j)}$ exists for all j

Example

T $C^\infty(R) \to C^\infty(R)$ (Not finite dimensional)

$T(f) = f'$

f is an eigenvector for T with eigenvalue λ if

$T(f) = \lambda f$ That is if f satisfies the

differential equation $f' = \lambda f$.

The solutions of this differential equation

are functions $f(x) = c\, e^{\lambda x}$

So every λ in R is an eigenvalue and the

associated eigenvectors are functions

$c\, e^{\lambda x}$ $c \neq 0$.

If $\lambda = 0$ the eigenvectors are non zero

constant functions.

Quiz T · F

Assume $T \in \mathcal{L}(V)$ $\dim V = n$

λ_1 and λ_2 are distinct eigenvalues of T

1. $E_{\lambda_1} + E_{\lambda_2}$ is direct
2. S indep T indep $S \cup T$ is indep
3. $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$
4. If T has n distinct eigenvalues then it is diagonalizable

$$a_1 v_1 + \cdots + a_n v_n + b_1 w_1 + \cdots + b_m w_m = 0$$

$$v_1 + v_2 = 0$$

Quiz

$N(T - \lambda I)$ $N(T - \lambda T)^2$

They are always equal

$\subseteq$

$\supseteq$