Lesson 17

Invariant subspaces

Eigenvalues and eigenvectors.

read 5.1

Def: A linear transformation

$T \quad V \to V$ is called a

(linear) operator

Def: A linear transformation
$T \quad V \to F$ is called

a (linear) functional

Def: A linear operator T on a finite dimensional vector space V is called diagonalizable if V has an ordered basis B s.t T_B^B is a diagonal matrix

Def A $n \times n$ matrix is called diagonalizable if
$$T : R^n \to R^n$$
$$Tx = Ax$$
is diagonalizable.

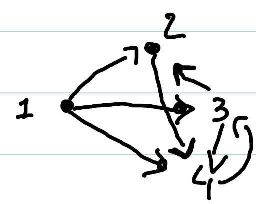
308 Connection

$A = T_E^E$ where E is the canonical basis

$$T_B^B = I_E^B \; T_E^E \; I_B^E$$
$$D = P^{-1} A P \quad \text{or}$$
$$P D P^{-1} = A$$

Useful to compute powers of A

1 Million doller eigenvector.
google algorithm to displey web pages from a search.



We can think of web pages as vertexes of a direct graph. The edges of the graph represent page links. We can represent this information using a matrix, where if a vertex han n outgoing edges, each vertex is given a weight of 1/n

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1 \\ 1/3 & 1 & 1/2 & 0 \end{pmatrix} \end{array} = A$$

Question: if I start from page 1, 2, 3, 4 with equal probability and I keep clicking at random one of the links I see on the page, after n steps what is the probability I am on page 4?

Answer involves computing powers of A.

Def: Suppose $T \in \mathcal{L}(V)$ and $U \leq V$. U is invariant under T if $T(U) \leq U$ i.e. $u \in U \Rightarrow T(u) \in U$

Ex 1: Given $T\ V \to V$, $N(T)$, $R(T)$, V, $\{0\}$ are all invariant under T,

Ex2: for a fixed $v \in V$

$$W = \text{span}\{T^n x \mid n \geq 0\} = \text{span}\{x, Tx, T^2x \ldots\}$$

is invariant under T

Note: if $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $T \in \mathcal{L}(V)$ and $T(U_i) \leq U_i$ then in order to understand T we just need to understand $T_{U_j} : U_j \to U_j$ for $j = 1 \ldots n$

The easiest situation is when each U_i has dimension 1

Ex3: Consider $T \; V \to V$

what are the dim 1 invariant subspaces ?

$U = \text{span}(v) \quad T(v) \in \text{Span}(v) \iff T(v) = \lambda v$

for some $\lambda \in F$

Def Given $T \in \mathcal{L}(V)$ a non zero vector s.t

$T(v) = \lambda v$ is called an eigenvector of T

λ is the associated eigenvalue

Ex 4 : given $T \in \mathcal{L}(P(\mathbb{R}))$ $T(p) = p'$

What are the eigenvalues and eigenvectors of T ?

If $T(p) = p' = \lambda p$ then p and p' are polynomials

of the same degree. This happens only if $p = c$ and $\lambda = 0$

Note if $T \in \mathcal{L}(V)$ and $\beta = \{v_1, v_2, \dots v_n\}$

is a basis for V then T_B^B is diagonal iff

each v_i is an eigenvector for T

in this case $V = \text{span}(v_1) \oplus \text{span}(v_2) \oplus \cdots \oplus \text{Span}(v_n)$

Def: If A is an $n \times n$ matrix eigenvectors eigenvalues of A are eigenvectors eigenvalues for L_A $R^n \to R^n$ $L_A(v) = Av$

i.e λ is an eigenvalue for A if there is a non zero vector v in R^n s.t $Av = \lambda v$. Such a v is called an eigenvector for A

Th 1: Given $T \in \mathcal{L}(V)$, where V is a finite dimensional vector space, the following are equivalent:

1) λ is an eigenvalue for T
2) $T - \lambda I$ is not injective :
3) $T - \lambda I$ is not surjective
4) $T - \lambda I$ is not invertible

Proof 1) $\Leftrightarrow$ 2) λ is an eigenvalue for T iff $T(v) = \lambda v$ for some $v \neq 0$ $v \in V$ that is iff $(T - \lambda I)v = 0$ iff $N(T-\lambda I) \neq \{0\}$ iff $T - \lambda I$ is not injective.

We already know 2) 3) 4) are equivalent (Using $\dim V = \dim N(T-\lambda I) + \dim R(T-\lambda I)$)

Note: eigenvectors for λ are the non zero vectors in $N(T-\lambda I)$

Thm If A is an $n \times n$ matrix
λ is an eigenvalue of A iff $\det(A-\lambda I)=0$

Proof assume $v \neq 0$ and $Av = \lambda I$
then $(A-\lambda I)v = 0$ so $A-\lambda I$ is
not invertible (308 stuff: column
of $A-\lambda I$ are dependent) so
$\det(A-\lambda I)=0$

vice-versa if $\det(A-\lambda I)=0$
then $A-\lambda I$ is singular and
therefore the system $(A-\lambda I)x = 0$ has
infinitely many solutions so
there must be some $v \neq 0$ s t
$(A-\lambda I)v = 0$ so $A = \lambda v$

Def $\operatorname{Det}(A - xI) = p(x)$
is called the characteristic polynomial
of A

Example:

$T \quad P_1(R) \longrightarrow P_1(R)$

$T(a+bx) = a+2b+3bx$

Compute the eigenvalues of T:

1. Choose some Basis B : $B = \{1, x\}$

2. $T_B^B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

3. Find the eigenvalues of B:

$\det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda)$

eigenvalues $\lambda = 1$, $\lambda = 3$

eigenvectors for $\lambda = 1$: solve

$\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ sol : $\{(x,0) \mid x \in R\}$

if $(1,0)$ is eigenvector for T_B^B

$p(x) = 1 + 0x$ is eigenvector for T

eigenvectors for $\lambda = 3$ solve

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

sol $\{ (x \ x) \mid x \in R \}$

if $(1, 1)$ is eigenvector for T_B^B

$p(x) = 1 \cdot 1 + 1 \cdot x$ is eigenvector for T