Lesson 16

Determinents

Ch 4

Notes on matrix multiplication

$$\underset{m\times n}{A}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 c_1 + x_2 c_2 + \cdots x_n c_n$$

$$(y_1\ y_2 \cdots y_m)\ A = y_1 r_1 + y_2 r_2 + \cdots + y_m r_m$$

$$A\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = c_1 \qquad (1 \cdots 0)A = r_1$$

$$\underset{m\times n}{A}\cdot\underset{n\times k}{B} \quad \begin{pmatrix} Ac_1^B & Ac_2^B & \cdots & Ac_n^B \end{pmatrix}$$

$$= \begin{pmatrix} r_1^A B \\ r_2^A B \\ r_n^A B \end{pmatrix}$$

Elementary matrices are matrics obtained performing 1 elementary operation on I

Performing elementary operations on a $n \times m$ matrix B corresponds to multiplying B by an elementary ($n \times n$) matrix

1) exchanging row i and row j

$$\begin{pmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \\ e_n \end{pmatrix} B$$

2) Multiplying row i by k

$$\begin{pmatrix} e_1 \\ \vdots \\ k e_i \\ \vdots \\ e_n \end{pmatrix} B$$

3) Replacing row i with row i + k row j

$$\begin{pmatrix} e_1 \\ \vdots \\ e_i + k e_j \\ \vdots \\ e_n \end{pmatrix} B$$

Th 1 : $\text{Det}(A\cdot B) = \text{Det}(A)\cdot \text{Det}(B)$

proof: If A is singuler then AB is singuler since the columns of AB are lineer combinations of columns of A (or by contrediction if AB is invertible then there is C s.t $(AB)\cdot C = I$ but then $A(BC) = I$ so A is invertible)

so $\det(AB) = \det(A)\cdot\det(B) = 0$

If A is not singuler we can find elementery metries. s.t $A = E_1 \cdots E_n I$ and we can prove by induction on n thet $\det(E_1 \cdots E_n B) = \det(E_1 \cdots E_n)\cdot \det(B)$

① Base case

a) E_1 is obtained from I by swepping two rows then $E_1 B$ is the metrix obtained by swepping the same 2 rows in B

$$\det(E_1 B) = -\det(B)$$
$$\det(E_1)\cdot\det(B) = -1\cdot\det(B)$$

b) E_1 is obtained by multiplying one row of I by $k (\neq 0)$
then $E_1 B$ is B with the same row multiplied by k

$$\det(E_1 B) = k\det(B)$$
$$\det(E_1)\cdot\det(B) = k\cdot\det(B)$$

c) E_1 is obtained by replacing row i of I with row i + k row s
then $E_1 B$ is the matrix obtained from B by replacing row i with row i + k row s

$$\det(E_1 B) = \det(B)$$
$$\det(E_1)\cdot\det(B) = 1\cdot\det(B)$$

Induction step:

Assume $\det(E_1 \cdots E_n B) = \det(E_1 \cdots E_n)\det(B)$
then $\det(E_1 \cdots E\, E_{n+1} B) = \det(E_1)\det(E_2 \cdots E_{n+1} B)$
$= \det(E_1)\det(E_2 \cdots E_{n+1})\det B = \det(E_1 E_2 \cdots E_{n+1})\det B$
$= \det(A)\det(B)$

Corollary : $\operatorname{Det}(A^{-1}) = \dfrac{1}{\operatorname{Det}(A)}$

therefore if A is non singular

$\det(A) \neq 0$.

Th 2: $\det(A^t) = \det(A)$

Proof : hw problem