Lesson 15

Determinants
ch 4

Determinants

We look for a function $D\ M_{n\times n} \to R$ s.t

D is lineer in each row, when the other rows are held fixed that is if $T: R^n \to R$ is defined by $T(\vec{x}) = D\begin{pmatrix} r_1 \to \\ r_2 \to \\ \vdots \\ x_i \to \\ \vdots \\ r_n \to \end{pmatrix}$ then

T is a lineer function. (Note: D itself will not be linear)

We also want $D(M)=0$ if M has two equal rows.

and $D(I) = 1$

Note: if $M = \begin{pmatrix} r_1 \to \\ \vdots \\ r_n \to \end{pmatrix}$ I will write

$D(r_1, r_2, \ldots r_n)$ instead of $D(M)$

Th1: Any such D has the following properties:

a) $D(r_1 \cdots r_i \cdots r_j \cdots r_n) = -D(r_1 \cdots r_j \cdots r_i \cdots r_n)$

b) $D(r_1 \cdots k r_i \cdots r_n) = k D(r_1 \cdots r_i \cdots r_n)$

c) $D(r_1 \cdots r_i + k r_j \cdots r_j \cdots r_n) = D(r_1 \cdots r_i \cdots r_j \cdots r_n)$

d) If M is singular i.e $r_1 \cdots r_n$ are dependent then $D(M) = 0$

e) If M is non singular then $D(M) \neq 0$

Proof:

a) $0 = D(r_1 \cdots r_i + r_j \cdots r_i + r_j \cdots r_n) = D(r_1 \cdots r_i \cdots r_i + r_j \cdots r_n)$
$+ D(r_1 \cdots r_j \cdots r_i + r_j \cdots r_n) = D(r_1 \cdots r_i \cdots r_i \cdots r_n)$
$+ D(r_1 \cdots r_i \cdots r_j \cdots r_n) + D(r_1 \cdots r_j \cdots r_i \cdots r_n)$
$+ D(r_1 \cdots r_j \cdots r_j \cdots r_n)$ Therefore
$0 = D(r_1 \cdots r_i \cdots r_j \cdots r_n) + D(r_1 \cdots r_j \cdots r_i \cdots r_n)$
so $D(r_1 \cdots r_i \cdots r_j \cdots r_n) = -D(r_1 \cdots r_j \cdots r_i \cdots r_n)$

b) This is one of the properties D is required to have

c) $D(r_1 \cdots r_i + k r_j \cdots r_j \cdots r_n) = D(r_1 \cdots r_i \cdots r_j \cdots r_n) +$
$D(r_1 \cdots k r_j \cdots r_j \cdots r_n) =$
$D(r_1 \cdots r_i \cdots r_j \cdots r_n) + k \underbrace{D(r_1 \cdots r_j \cdots r_j \cdots r_n)}_{0}$

d) In this case one row is a linear combination of the others. We can assume the last row $r_n = a_1 r_1 + \cdots + a_{n-1} r_{n-1}$ (a similar argument will work if any other row is a linear combination of the other rows)

$D(r_1 \cdots r_{n-1}\ a_1 r_1 + \cdots a_{n-1} r_{n-1}) = a_1 D(r_1 \cdots r_{n-1}\ r_1)$
$+ a_2 D(r_1\ r_2 \cdots r_{n-1}\ r_2) + \cdots + a_{n-1} D(r_1 \cdots r_{n-1}\ r_{n-1})$
$= 0$

e) In this case a sequence of elementary operations i.e replacing some row r_i with $r_i + k r_j$ or multiplying a row by a non zero scalar k or swapping rows can reduce M to I, therefore

$1 = \det(I) = c \det(M)$ where $c \neq 0$

Th 2 (Uniqueness): Suppose $D_1 \; M_{n\times n} \to R$ and $D_2 \; M_{n\times n} \to R$ both have the required properties and $M \in M_{n\times n}$.

If M is singular then $D_1(M) = D_2(M) = 0$

If not there is a sequence of elementary operations that transform I into M

$$I \to M_1 \to M_2 \cdots\cdots M_s = M$$

and since $D_1(I) = D_2(I)$ then $D_1(M_1) = D_2(M_2)$ and $D_1(M_2) = D_2(M) \;\cdots\cdots\; D_1(M) = D_2(M)$

Th 3 (Existence) define D by recursion:

$D_1 \quad M_{1\times 1} \longrightarrow R \qquad D_1(a) = a$

$D_n \quad M_{n\times n} \longrightarrow R$

$$D_n(M) = \sum_{i=1} (-1)^{1+j} M_{1j} \, D_{n-1}(M_{1j})$$

Where M_{1j} is the matrix we obtain from M by deleting the 1^{st} row and j^{th} column.

The determinant function from 308 has the desired properties.

No proof. (See th 4.3 of the textbook for a partial proof).

Th4: you can expand along any row,

that is for $1 \le k \le n$

$$D_n(M) = \sum_{i=1} (-1)^{k+j} M_{kj} D_{n-1}(M_{kj})$$

No proof