

Lesson 14

Linear transformations

Change of coordinates

Textbook 2.5

Th 1: if V is a finite dimensional vector space and $B_1 = \{v_1, \dots, v_n\}$ is a basis for V then

$$\varphi: V \longrightarrow \mathbb{R}^n$$
$$\varphi(v) = [v]_{B_1}$$

is an isomorphism.

Proof: Let $e_1 = (1, \dots, 0)$ $e_2 = (0, 1, \dots, 0) \dots$ in \mathbb{R}^n and let $B_2 = \{e_1, e_2, \dots, e_n\}$ then $\{\varphi(v_1), \varphi(v_2), \dots, \varphi(v_n)\} = B_2$ is a basis for \mathbb{R}^n

Note: remember isomorphisms map linearly independent sets to linearly independent sets, spanning sets to spanning sets, bases to bases

Th 2: If $\dim V = n$ and $\dim W = m$
 $\mathcal{L}(V, W) \cong M_{m \times n}(F)$ therefore

$$\dim(\mathcal{L}(V, W)) = m \times n$$

Proof: Fix bases $B_1 = \{v_1, v_2, \dots, v_n\}$ in V
and $B_2 = \{w_1, w_2, \dots, w_m\}$ in W and

consider $\varphi: \mathcal{L}(V, W) \rightarrow M_{m \times n}$
 $\varphi(T) = T_{B_1}^{B_2}$

We want to show φ is an isomorphism.

1) It is a linear transformation
proved in Lesson 11

2) It is 1-1 since $\varphi(T) = 0$
means $T_{B_1}^{B_2} = 0$ so $T = 0$

3) It is onto since given $M = (a_{ij})$ in $M_{m \times n}$

$M = \varphi(T) = T_{B_1}^{B_2}$ for the linear
transformation T such that

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m$$

for $j=1, \dots, n$

Def $L(V, F)$ is called

the dual space of V

often denoted by V^*

HW problem: Find a basis for $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$

$$\textcircled{1} \quad \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \cong M_{3 \times 2}$$

Basis for $M_{3 \times 2}$ is :

$$\pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \pi_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \pi_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \pi_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\pi_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \pi_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\textcircled{2}$ Fix the bases $B_1 = \{(1,0), (0,1)\}$
 $B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$ in \mathbb{R}^3 . Consider the isomorphism $\varphi: \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \rightarrow M_{3 \times 2}$ $\varphi(T) = T_{B_1}^{B_2}$
find

linear transformations $T_1, T_2, T_3, T_4, T_5, T_6$
s.t $\varphi(T_i) = \pi_i$

In other words you want to
think: if $M_C = T_{B_1}^{B_2}$ what is T ?

For example if

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = T_{B_1}^{B_2} = \left[\begin{array}{c} [T(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})]_{B_2} \\ [T(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})]_{B_2} \end{array} \right]$$

What is T ?

$$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot (100) + 0(010) + 0(001) \\ = (100)$$

$$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0(100) + 0(010) + \\ + 0(001) = (0, 0, 0)$$

$$\text{so } T(x \ y) = T(x(10) + y(01))$$

$$= x T(10) + y T(01) =$$

$$= x(100) + y(000) = (x \ 00)$$

Matrix of change of basis

Def: $I: V \rightarrow V$ is the linear transformation defined by $I(v) = v$.

Def: Given bases B_1 and B_2 in V , we call $I_{B_1}^{B_2}$ the matrix of change of basis from B_1 to B_2 . If $B_1 = v_1, v_2, \dots, v_n$

$$I_{B_1}^{B_2} = \begin{bmatrix} [v_1]_{B_2} & \dots & [v_n]_{B_2} \\ \vdots & & \vdots \end{bmatrix}$$

$$I_{B_1}^{B_2} [v]_{B_1} = [v]_{B_2}$$

$$\text{Th 3: } I_{B_2}^{B_1} = \left(I_{B_1}^{B_2} \right)^{-1}$$

$$\text{Proof } I_{B_2}^{B_1} \cdot I_{B_1}^{B_2} = I_{B_1}^{B_1} = I$$

Th 4: If $T \in \mathcal{L}(V)$ and B_1 and B_2 are bases for V then

$$T_{B_2}^{B_2} = I_{B_2}^{B_1} \cdot T_{B_1}^{B_1} \cdot I_{B_1}^{B_2}$$

Proof $I_{B_2}^{B_1} T_{B_1}^{B_1} I_{B_1}^{B_2} [v]_{B_2} =$

$$I_{B_2}^{B_1} T_{B_1}^{B_1} [v]_{B_1} =$$

$$I_{B_2}^{B_1} [T v]_{B_1} =$$

$$[T(v)]_{B_2}$$

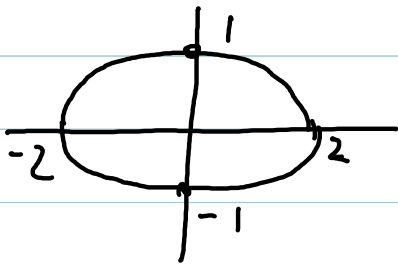
So For the example in video 12

how are

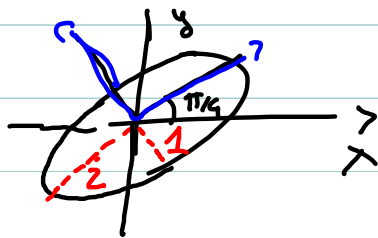
$$T_B^B \quad T_B^{B_1} \quad T_{B_1}^{B_1}$$

related?

Application:



I want to rotate it counterclockwise of an angle of $\frac{\pi}{4}$

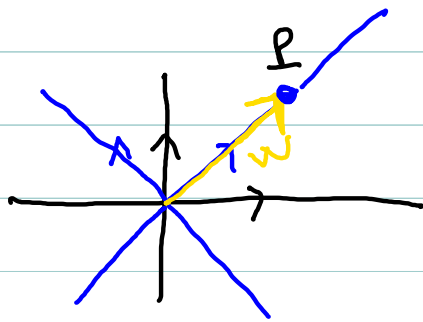


Find equation in x, y plane

$$\frac{x'^2}{2^2} + \frac{y'^2}{1} = 1 \quad (x', y') \text{ coordinates with respect}$$

to Basis $B_2 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

want to use Basis $E = \{(1,0), (0,1)\}$ instead



In basis B_2
 w has coordinates $(2, 0)$ P has
coordinates $x^1 = 2$
 $y^1 = 0$

In basis E
 w has coordinates

$\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right)$, P has

coordinates $x = \frac{2}{\sqrt{2}}$ $y = \frac{2}{\sqrt{2}}$

To go from an equation in $x' y'$
to an equation in $x y$ we

need to express $x' y'$ in

terms of x and y so

we use $I_E^{B_2}$ since

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = I_E^{B_2} \begin{pmatrix} x \\ y \end{pmatrix} \quad \bullet \quad \text{How do we find}$$

$$I_E^{B_2} \quad ? \quad I_E^{B_2} = \begin{bmatrix} [(1,0)]_{B_2} & [(0,1)]_{B_2} \\ & \end{bmatrix}$$

so we need to figure out $[(1,0)]_{B_2}$ and

$[(0,1)]_{B_2}$ or we can use the fact

that

$$I_E^{B_2} = (I_{B_2}^E)^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = 1/\sqrt{2} x + 1/\sqrt{2} y$$

$$y' = -1/\sqrt{2} x + 1/\sqrt{2} y$$

$$\frac{1}{4} x'^2 + y'^2 = 1$$

becomes

$$\frac{1}{4} \left(\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \right)^2 + \left(-\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \right)^2 = 1$$

$$\frac{1}{4} \frac{1}{2} x^2 + \frac{1}{4} \frac{1}{2} y^2 + \frac{1}{2} \frac{1}{2} xy +$$

$$\frac{1}{2} x^2 + \frac{1}{2} y^2 - xy = 1$$

$$\frac{5}{8} x^2 + \frac{5}{8} y^2 - \frac{3}{4} xy = 1$$