Lesson 13

Inverses - Isomorphism

Textbook 2.4

Def Given $T\ V \to W$, T^{-1} the inverse of T (if it exists) is the function $T^{-1}\ W \to V$ s.t

$T(v) = w \iff T^{-1}(w) = v$ i.e $T \circ T^{-1} = I_W$ $T^{-1} \circ T = I_W$

Th1: T^{-1} exists if and only if T is one to one and onto.

In this case we say T is invertible.

Th2: If $T\ V \to W$ is an invertible linear transformation then $T^{-1}\ W \to V$ is a linear transformation.

Proof: We need to show $T^{-1}(w_1+w_2) = T(w_1) + T(w_2)$ and $T^{-1}(Kw_1) = KT^{-1}(w_1)$ for all $w_1, w_2 \in W$, $K \in F$. Let

$T(v_1) = w_1$ then $T^{-1}(w_1) = v_1$

$T(v_2) = w_2$ then $T^{-1}(w_2) = v_1$

Then $T(v_1+v_2) = T(v_1) + T(v_2) = w_1 + w_2$

Therefore $T^{-1}(w_1+w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$

and $T(Kv_1) = K\,T(v_1) = kw_1$

Therefore $T^{-1}(Kw_1) = kv_1 = kT^{-1}(w_1)$

Th3: If V is finite dim then $T\ V \to W$ is invertible $\Leftrightarrow$ $\dim R(T) = \dim V = \dim W$

Proof: Assume T is invertible, then T is 1-1 and onto therefore $V = \dim(N(T)) + \dim(R(T))$ gives us $V = \dim(R(T)) = \dim W$

Vice-versa assume

$$\dim R(T) = \dim W = \dim V$$

then T is onto and using $V = \dim(N(T)) + \dim(R(T))$ we get $\dim N(T) = 0$ so T is 1-1

Therefore T is invertible.

Th 4: Let V, W be finite dimensional

Let $T: V \to W$ be a linear transformation and let B_1 and B_2 be ordered bases in V and W; then T is invertible iff $T_{B_1}^{B_2}$ is invertible, and $T^{-1}{}_{B_2}^{B_1} = (T_{B_1}^{B_2})^{-1}$

Proof: suppose T is invertible, then $\dim V = \dim W$ by previous th so $T_{B_1}^{B_2}$ is a square matrix and $T \circ T^{-1} = T^{-1} \circ T = I$ therefore

$$T_{B_1}^{B_2} \cdot T^{-1}{}_{B_2}^{B_1} = I_{B_2}^{B_2} = \underline{I} = I_{B_1}^{B_1} = T^{-1}{}_{B_2}^{B_1} \cdot T_{B_1}^{B_2}$$

so $(T_{B_1}^{B_2})^{-1} = T^{-1}{}_{B_2}^{B'}$

Viceverse suppose $T_{B_1}^{B_2}$ is invertible and M is its inverse define $S: W \to V$ in the following way $M \begin{bmatrix} | & | & | \end{bmatrix}$ ith column of M is $[T^{-1}(w_i)]_{B_1}$

So $M = S_{B_2}^{B_1}$. We want to show $S = T^{-1}$ that is we want to show $\forall v \in V$ $S \circ T(v) = v$ $\forall w \in W$ $T \circ S(w) = w$. We can compute using matrices:

$\vdots$

given $v \in V$ compute

$$[S \circ T(v)]_{B_1} = \underset{\underset{I}{\shortparallel}}{S^{B_1}_{B_2}\, T^{B_2}_{B_1}} [v]_{B_1} = I\,[v]_{B_1} = [v]_{B_1}$$

so $S \circ T(v) = v$

given $w \in W$

$$[T \circ S(w)]_{B_2} = T^{B_2}_{B_1}\, S^{B_1}_{B_2}\, [w]_{B_2} = I\,[w]_{B_2}$$

$$= [w]_{B_2}$$

so $T \circ S(w) = w$

Def If $T\ V \to W$ is an invertible linear transformation we call T an ISOMORPHISM and we say V and W are isomorphic, and we write $V \approx W$

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Th 5: Let $T\ V \to W$ be an isomorphism.
$v_1\ v_2 \cdots v_n$ are linearly independent in $V \Leftrightarrow$
$T(v_1)\ T(v_2) \cdots T(v_n)$ are linearly independent
in W.

Proof: Let T be an isomorphism.

suppose that $v_1\ v_2 \cdots v_n$ are linearly independent
and consider $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0$
$T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0$ T is 1-1 so
$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ so $a_1 = a_2 = \cdots = a_n = 0$

Now suppose $T(v_1), T(v_2), \cdots T(v_n)$ are
linearly independent and that
$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ Then
$T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0$
$a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0$
so $a_1 = a_2 = \cdots = a_n = 0$

Th 6: If $T\ V \to W$ is an isomorphism, then $\{v_1, v_2, \cdots v_n\}$ spans V $\Leftrightarrow$ $\{T(v_1), T(v_2), \cdots T(v_n)\}$ spans W

Proof Let T be an isomorphism.

Assume $\{v_1 \dots v_n\}$ spans V and let $w \in W$, since T is onto, $W = R(T)$ so $w = T(v)$ for some $v \in V$ and $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some scalers $\alpha_1 \dots \alpha_n$, so $w = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$

Assume $\{T(v_1), T(v_2), \quad, T(v_n)\}$ spans W and $v \in V$ $T(v) = \alpha_1 T(v_1) + \cdot + \alpha_n T(v_n)$ for some scalers $\alpha_1, \alpha_2, \cdots \alpha_n$ so $T(v) = T(\alpha_1 v_1 + \cdots \alpha_n v_n)$; since T is 1-1 $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Therefore we have

Th 7: If $T\ V \to W$ is an isomorphism $B_1 = \{v_1, v_2, \dots, v_n\}$ is a basis for V $\Leftrightarrow$ $\{T(v_1), \dots T(v_n)\}$ is a basis for W

Th 8: If $T \ V \to W$ is a linear transformation and $B_1 = \{v_1, v_2, \dots v_n\}$ is a basis for V and $B_2 = \{T(v_1) \ T(v_2) \cdots T(v_n)\}$ Then T is an isomorfism iff B_2 is a basis for W

Proof: If T is an isomorphism then the previous 2 theorems prove B_2 is a basis for W

Vice-verse assume that B_2 is a basis for W and consider

$$T_{B_1}^{B_2} = \begin{bmatrix} [T(v_1)]_{B_2} & [T(v_2)]_{B_2} & \cdots & [T(v_n)]_{B_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

which is invertible, so T is invertible.