Lesson 12

One example

Show there is a unique linear transformation $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that $T(1+x+x^2)=2$, $T(4-2x^2)=x^2$, $T(2x)=x+5x^2$.

Compute $T(x+1)$.

We just need to show $1+x+x^2$, $4-2x^2$, $2x$ is a basis for $P_2(\mathbb{R})$. Let' check that $\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}4\\0\\-2\end{pmatrix}\begin{pmatrix}0\\2\\0\end{pmatrix}$ is a basis for $\mathbb{R}^3$

reduce $\begin{pmatrix}1 & 4 & 0\\1 & 0 & 2\\1 & -2 & 0\end{pmatrix}$ to echelon form

$\begin{pmatrix}1 & 4 & 0\\0 & -4 & 2\\0 & 0 & -3\end{pmatrix}$ pivot in every row column. Yes it is a basis

Since given a basis $v_1 \dots v_n$ for a vector space V and n wectors

$w_1, w_2, \dots w_n$ in a vector space W, there is a unique linear transformation $T\ V \to W$ such that $T(v_i) = w_i$

we know there is a unique linear transformation $T\ P_2(R) \to P_2(R)$ such that $T(1+x+x^2) = 2$ $T(4-2x^2) = x^2$ $T(2x) = x+5x^2$.

To compute $T(1+x)$

Let $B = 1+x+x^2,\ 4-2x^2,\ 2x$

Then $T_B^B = \begin{bmatrix} [T(1+x+x^2)]_B & [T(4-2x^2)]_B & [T(2x)]_B \\ & & \end{bmatrix}$

$T(1+x+x^2) = 2$; $[2]_B = (a, b, c)$ if

$2 = a(1+x+x^2) + b(4-2x^2) + c(2x)$

To find a, b, c let's use R^3 again

Finding (a, b, c) corresponds to solving the vector equation

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

That is the system with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right)$$

Let's work more in general with $\left(\begin{array}{ccc|c} 1 & 4 & 0 & z_1 \\ 1 & 0 & 2 & z_2 \\ 1 & -2 & 0 & z_3 \end{array}\right)$ and reduce it to echelon form:

$$\left(\begin{array}{ccc|c} 1 & 4 & 0 & z_1 \\ 0 & -4 & 2 & z_2 - z_1 \\ 0 & 0 & -3 & z_3 + \frac{z_1}{2} - \frac{3z_2}{2} \end{array}\right) \quad (*)$$

if $z_1 = 2$, $z_2 = 0$, $z_3 = 0$

we get $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 2 \\ 0 & -4 & 2 & -2 \\ 0 & 0 & -3 & 1 \end{array}\right)$

which has solutions : $x_3 = -\frac{1}{3}$

$x_2 = \frac{1}{3}$, $x_1 = \frac{2}{3}$ so

$$2 = \frac{2}{3}(1 + x + x^2) + \frac{1}{3}(4 - 2x^2) - \frac{1}{3}(2x)$$

$[2]_B = \left(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right)$

In a similar way to find

$[T(4-2x^2)]_B = [x^2]_B$ use $z_1=0$, $z_2=0$

$z_3=1$ and solve $(*)$

$x_1 = \frac{2}{3}$ $x_2 = -\frac{1}{6}$, $x_3 = -\frac{1}{3}$

$[x^2]_B = \left(\frac{2}{3}, -\frac{1}{6}, -\frac{1}{3}\right)$

and $[T(2x)]_B = [x+5x^2] = \left(\frac{10}{3}, \frac{-5}{6}, \frac{-7}{6}\right)$

so $$T_B^B = \begin{bmatrix} 2/3 & 2/3 & 10/3 \\ 1/3 & -1/6 & -5/6 \\ -1/3 & -1/3 & -7/6 \end{bmatrix}$$

Finally $[1+x]_B = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}\right)$

so to find $T(1+x)$

① Multiply

$$\begin{bmatrix} 2/3 & 2/3 & 10/3 \\ 1/3 & -1/6 & -5/6 \\ -1/3 & -1/3 & -7/6 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{13}{9} \\ -\frac{7}{36} \\ -5/9 \end{bmatrix}$$

② $T(x+1) = \frac{13}{9}(1+x+x^2) - \frac{7}{36}(4-2x^2)$

$-\frac{5}{9}(2x) = \frac{11}{6}x^2 + \frac{1}{3}x + \frac{2}{3}$

This was a lot of calculations!
can we do something simpler?

How about using $B_1 = \{1, x, x^2\}$
and computing $T_B^{B_1} =$

$$= \begin{bmatrix} [T(1+x+x^2)]_{B_1} & [T(4-2x^2)]_{B_1} & [T(2x)]_{B_1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$[T(1+x)]_B = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}\right)$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{pmatrix} 1/3 \\ 1/6 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 11/6 \end{pmatrix}$$

$$T(1+x) = \frac{2}{3} + \frac{1}{3}x + \frac{11}{6}x^2$$

Much easier !

What would $T_{B_1}^{B_1}$ be ?

Is there a relationship between

T_B^B $\quad T_B^{B_1}$ $\quad T_{B_1}^{B_1}$?