Lesson 11

Algebra of linear transformations

Textbook 2.3

Algebra of linear transformations

Let V and W be vector spaces over F

Def: Let $\mathcal{L}(V, W)$ be the set of all linear transformations from V to W

Sometimes we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, V)$.

Troughout this lesson, if we need to assume V, W are finite dimensional, then we assume

$B_1 = \{v_1, v_2, \dots v_n\}$ is an ordered basis for V and $B_2 = \{w_1, w_2, \dots w_m\}$ is an ordered basis for W.

Our first goal is to define + and scalar multiplication on $\mathcal{L}(V, W)$

Def Let $T \in \mathcal{L}(V,W)$, let $k \in F$ and $kT : V \to W$ be defined by

$(kT)(v) = k\,T(v)$ then

Th1: kT_L is a linear transformation.

Proof: if $v \in V$, and $h \in F$ then

$(kT)(hv) = k\left(T(hv)\right) =$

$= k(hT(v)) = h\left(kT(v)\right) = h(kT)(v)$

if $v_1, v_2 \in V$ $(kT)(v_1+v_2) = k\left(T(v_1+v_2)\right)$

$= k\left(T(v_1)+T(v_2)\right) = kT(v_1) + kT(v_2) =$

$(kT)(v_1) + (kT)(v_2)$

Def: If $T_1, T_2 \in \mathcal{L}(V, W)$ let $T_1+T_2 : V \to W$ be defined by $(T_1+T_2)(v) = T_1(v)+T_2(v)$

Th 2: $T_1+T_2 \quad V \to W$ is a linear transformation.

Hw problem

Th3: $\mathcal{L}(V\ W)$ is a vector space over F

1) $+$ is commutative i.e $T+S=S+T$ for all S, T in $\mathcal{L}(V,W)$

2) $+$ is associative, i.e $(A+B)+C=(A+B)+C$ for all A, B, C in $\mathcal{L}(V,W)$

3) $0: V\to W$, $0(v)=0$ is the "0 element in $\mathcal{L}(V,W)$ i.e $0+T=T+0$ for all T in $\mathcal{L}(V,W)$

4) Given $T\in\mathcal{L}(V,W)$ $-T$ is the transformation $(-T)\ V\to W$ $(-T)(v)=-T(v)$

5) $1\cdot T=T$ for all T in $\mathcal{L}(V,W)$

6) $(a\cdot b)T=a\cdot(b\cdot T)$ for all a, b in F and T in $\mathcal{L}(V,W)$

7) $a(T+S)=aT+aS$ for all a in F and T, S in $\mathcal{L}(V,W)$

8) $(a+b)T=aT+bT$ for a, b in F, T in $\mathcal{L}(V,W)$

Idea: if $\dim V = n$ and $\dim W = m$ and we fix ordered bases B_1 in V and B_2 in W. Every $T \in \mathcal{L}(V, W)$ corresponds to a $m \times n$ matrix $T_{B_1}^{B_2}$

We want to argue that
$\mathcal{L}(V, W)$ is "the same"
as $M_{m \times n}(F)$

The next theorem shows

$$\varphi : \mathcal{L}(V, W) \longrightarrow M_{m \times n}(F)$$

$$\varphi(T) = T_{B_1}^{B_2}$$

is a linear transformation

Recall $[\ ]_{B_2} : W \longrightarrow R^m$
$w \to [w]_{B_2}$

is also a linear transformation

Th 4: If V and W are finite dimensional and B_1 is an ordered basis for V and B_2 is an ordered basis for W then

$$(T_1+T_2)_{B_1}^{B_2} = T_1{}_{B_1}^{B_2} + T_2{}_{B_1}^{B_2}$$

and

$$(K T_1)_{B_1}^{B_2} = K\left(T_1{}_{B_1}^{B_2}\right)$$

Proof: Let $v \in V$: $\left(T_1{}_{B_1}^{B_2} + T_2{}_{B_1}^{B_2}\right)[v]_{B_1} =$

$$T_1{}_{B_1}^{B_2}[v]_{B_1} + T_2{}_{B_1}^{B_2}[v]_{B_1} = [T_1(v)]_{B_2} + [T_2(v)]_{B_2} =$$

$$= [T_1(v)+T_2(v)]_{B_2} = [(T_1+T_2)(v)]_{B_2}$$

(This is just saying that if $B_2 = \{w_1, \dots w_n\}$ and $T_1(v) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$ and $T_2(v) = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n$ then $T_1(v) + T_2(v) = (\alpha_1+\beta_1) w_1 + (\alpha_2+\beta_2) w_2 + \cdots$ $\cdots + (\alpha_n+\beta_n) w_n$)

$$\left(k\, T_{1\,B_1}^{\;B_2}\right)[v]_{B_1} = k\left(T_{B_1}^{B_2}[v]_{B_1}\right) = k[T(v)]$$

$$= [k\,T(v)]_{B_2} = [T(kv)]_{B_2} \qquad B_2$$

Th 5: $\varphi \quad \mathcal{L}(V,W) \rightarrow M_{m\times n}(F)$

$$\varphi(T) = T_{B_1}^{B_2}$$

is 1-1 and onto

Proof: Suppose $B_1 = \{v_1 \cdots v_n\}$
$B_2 = \{w_1 \cdots w_m\}$

1) If $\varphi(T) = T_{B_1}^{B_2} = 0$ then
$T(v_1) = T(v_2) \cdots = T(v_n) = 0$
therefore T is the 0 transformatio
so the $N(\varphi) = \{0\}$ and therefore
φ is 1-1.

2) To show φ is onto, take $M \in M_{m\times n}$
$M = (c_{ij})$: and define $T: V \rightarrow W$ by

$T(v_1) = c_{11}w_1 + c_{21}w_2 + \cdots + c_{m1}w_m$
$T(v_2) = c_{12}w_1 + c_{22}w_2 + \quad + c_{m2}w_m \quad \cdots$

Then $T_{B_1}^{B_2} = M$

Th 6: Let $T\ V \to W$, $S\ W \to U$ be linear tranformations; then ST (the composition of S and T) $V \to U$ is a linear transformation.

$ST(v) = S(T(v))$

Proof: $ST(v_1+v_2) = S(T(v_1+v_2)) = S(T(v_1)+T(v_2))$

$= ST(v_1) + ST(v_2)$

$ST(kv) = S(kT(v)) = kST(v)$

$S(T(kv))$

Th 7: Let V, W, U be finite dimensional and Let $\beta_1, \beta_2, \beta_3$ be ordered bases for V, W, U

then $(ST)_{\beta_1}^{\beta_3} = S_{\beta_2}^{\beta_3} T_{\beta_1}^{\beta_2}$

Proof: if $v \in V$ $\quad S_{\beta_2}^{\beta_3} \cdot T_{\beta_1}^{\beta_2} [v]_{\beta_1} = S_{\beta_2}^{\beta_3} [T(v)]_{\beta_2}$

$= [ST(v)]_{\beta_3}$ $\quad$ therefore $S_{\beta_2}^{\beta_3} T_{\beta_1}^{\beta_2} = (ST)_{\beta_1}^{\beta_3}$