

Lesson 10

Linear transformations

Matrix representation

Textbook: 2.2

Def An ordered basis for a finite dimensional vector space V is a basis endowed with a specific order.

Def Given a vector space V , an ordered basis $B = \{v_1, \dots, v_n\}$ for V and a vector $v \in V$ we define the coordinate vector of v relative to B , $[v]_B$ as the unique vector $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ s.t. $v = d_1 v_1 + \dots + d_n v_n$

Note: this allows us to think of a vector in a space of dimension n as an element of \mathbb{R}^n

Ex 1: Take $P_3(\mathbb{R})$ $B = \{1, x, x^2, x^3\}$

$$[x^3 - x + 5]_B = (5, -1, 0, 1)$$

Th1: Given a finite dimensional
vector space V and an ordered basis
 $B_1 = \{v_1, v_2, \dots, v_n\}$ for V

$$T: V \longrightarrow \mathbb{R}^n$$

$$T(v) = [v]_{B_1}$$

is a linear transformation and
it is injective and surjective.

Proof: suppose $v, w \in V$

$$v = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$$

$$w = h_1 v_1 + h_2 v_2 + \dots + h_n v_n$$

$$\text{then } v+w = (k_1+h_1)v_1 + \dots + (k_n+h_n)v_n$$

$$\Delta v = \Delta k_1 v_1 + \dots + \Delta k_n v_n$$

$$T(v+w) = (k_1+h_1, k_2+h_2, \dots, k_n+h_n) =$$

$$(k_1, \dots, k_n) + (h_1, \dots, h_n) = T(v) + T(w)$$

$$T(\Delta v) = (\Delta k_1, \dots, \Delta k_n) = \Delta(k_1, \dots, k_n) = \Delta T(v)$$

T is surjective (onto): given any vector

$$(y_1, \dots, y_n) \text{ in } \mathbb{R}^n \text{ take } v = y_1 v_1 + \dots + y_n v_n$$

$$\text{Then } T(v) = (y_1, \dots, y_n)$$

T is injective (one to one): if $T(v) = T(w) = (x_1, \dots, x_n)$

$$\text{Then } v = w = x_1 v_1 + \dots + x_n v_n$$

Th2: Given a basis $B = \{v_1, v_2, \dots, v_n\}$ in V and n vectors $\{w_1, \dots, w_n\}$ in W

There is a unique linear transformation $T: V \rightarrow W$ s.t. $T(v_i) = w_i$

Proof: let $v \in V$ then v can be written in a unique way as a linear combination of B

$$v = \sum_{i=1}^n \alpha_i v_i \quad \text{so we must define} \quad T(v) = \sum_{i=1}^n \alpha_i w_i$$

Is this a linear transformation? If $u \in V$

$$u = \sum_{i=1}^n \alpha_i v_i \quad \text{then}$$

$$T(\lambda v) = \sum_{i=1}^n \lambda \alpha_i w_i = \lambda \sum_{i=1}^n \alpha_i w_i = \lambda T(v)$$

$$T(v+u) = \sum_{i=1}^n (\alpha_i + \beta_i) w_i = \sum_{i=1}^n \alpha_i w_i + \sum_{i=1}^n \beta_i w_i = T(v) + T(u)$$

So yes it is a linear transformation

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$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$B_1 = \{ e_1, e_2, e_3 \} \quad \text{i.e. } \left\{ \begin{matrix} (1,0,0) & (0,1,0) \\ & (0,0,1) \end{matrix} \right\}$$

$$B_2 = \{ e_1, e_2 \} \quad \text{i.e. } \{ (1,0) \ (0,1) \}$$

The matrix of T is

$$M = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$\text{Then } T(v) = Mv$$

why? If $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined
by $Sv = Mv$

$$M \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \text{first column of } M \quad \text{so } Se_1 = Te_1$$

$$M \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \text{second column of } M \quad \text{so } Se_2 = Te_2$$

$$M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \text{third column of } M \quad \text{so } Se_3 = Te_3$$

Therefore $S = T$

Matrices of linear transformation

Given two finite dim spaces V and W with ordered bases $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{w_1, \dots, w_m\}$ and a linear transformation $T: V \rightarrow W$

there exists a unique matrix $T_{B_2}^{B_1}$

$$\text{s.t. } \underbrace{[T[v]]}_{\text{in } \mathbb{R}^m} = T_{B_2}^{B_1} \underbrace{[v]_{B_1}}_{\text{in } \mathbb{R}^n}$$

Let's look at one example before we prove this theorem

$$\text{Ex 2 } T: P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$$

$$T(p) = p'$$

If we choose bases $B_1 = \{1, x, x^2\}$ in $P_2(\mathbb{R})$ and $\{1, x\}$ in $P_1(\mathbb{R})$ what is $T_{B_1}^{B_2}$?

IDEA: Given a polynomial $p(x) = a \cdot 1 + b \cdot x + c \cdot x^2$ look at $[p]_{B_1} = (a, b, c) \in \mathbb{R}^3$ instead, and

$T(p) = b + 2cx$ so $[T(p)]_{B_2} = (b, 2c) \in \mathbb{R}^2$. Look at the linear transformation $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ instead $S(a, b, c) = (b, 2c)$

$(S: (a, b, c) \rightarrow p(x) = a + bx + cx^2 \rightarrow T(p) = b + 2cx \rightarrow (b, 2c))$
composition of linear transformations

$$S_{\{e_1, e_2, e_3\}}^{\{e_1, e_2\}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = T_{B_1}^{B_2}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$$

Alternative way to find $T_{B_1}^{B_2}$

① Calculate $T(1) = 0$

$$T(x) = 1$$

$$T(x^2) = 2x$$

② Find coordinates in B_2

$$0 = 0 \cdot 1 + 0 \cdot x \quad \text{so } [0]_{B_2} = (0, 0)$$

$$1 = 1 \cdot 1 + 0 \cdot x \quad \text{so } [1]_{B_2} = (1, 0)$$

$$2x = 0 \cdot 1 + 2 \cdot x \quad \text{so } [2x]_{B_2} = (0, 2)$$

$$T_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

In general $T_{B_1}^{B_2} = \begin{bmatrix} [T(B_1)]_{B_2} & \dots & [T(B_n)]_{B_2} \\ \downarrow & & \downarrow \end{bmatrix}$

Recall: $T: V \rightarrow W$ $\{v_1, \dots, v_n\}$
 B_1, B_2 are ordered

bases in V and W . We want a matrix

$$T_{B_1}^{B_2} \text{ s.t. } T_{B_1}^{B_2} [v]_{B_1} = [T(v)]_{B_2}$$

$$\text{Proof: } T_{B_1}^{B_2} = \begin{bmatrix} [T(v_1)]_{B_2} & [T(v_n)]_{B_2} \\ \vdots & \vdots \end{bmatrix}$$

We need to check that $T_{B_1}^{B_2} [v]_{B_1} = [T(v)]_{B_2}$
for all $v \in V$

$$\cdot \text{ Given } v = \alpha_1 v_1 + \dots + \alpha_n v_n, \quad [v]_{B_1} = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$T(v) = \alpha_1 T(v_1) + \dots + \alpha_n T(v_n) \in W$$

$$[T(v)]_{B_2} = \alpha_1 [T(v_1)]_{B_2} + \dots + \alpha_n [T(v_n)]_{B_2}$$

$$= T_{B_1}^{B_2} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Def: $M_{m \times n}(F)$ is the set of all $m \times n$ matrices with entries in F .

Th 3: $M_{m \times n}(F)$ is a vector space of dimension $m \times n$

Proof: from hw 1 you know it is a vector space.

A basis is

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \dots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & & & 0 \end{bmatrix}$$

$$\dots \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}$$