Lesson 1

Vector spaces and Subspaces.

Textbook: 1.1, 1.2

Def: a **field** is a set S, together with operations $+, \cdot$
and special elements $0, 1$ ($0 \neq 1$) such that

1) $+$ is commutative i.e $x + y = y + x$ for all x, y in S

2) $+$ is associative i.e $(x + y) + z = x + (y + z)$ for all x, y, z in S

3) 0 is the identity for $+$ i.e $x + 0 = 0 + x = x$ for all x in S

4) Every element x in S has an additive inverse $(-x)$ in S s.t $x + (-x) = (-x) + x = 0$

5) $\cdot$ is commutative

6) $\cdot$ is associative

7) 1 is the identity for $\cdot$, that is $1 \cdot x = x \cdot 1 = x$ $\forall x \in S$

8) Every $x \neq 0$ in S has a multiplicative inverse x^{-1}. $x^{-1} \in S$ s.t
$x \cdot x^{-1} = x^{-1} \cdot x = 1$

9) Multiplication distributes over addition i.e $x(y + z) = xy + x \cdot z$
for all x, y, z in S

Ex: $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_2$

(Note:) In this class when we talk about a field F
we mean $\mathbb{R}$ or $\mathbb{C}$

Def: a **Vector space** over a field F is a set V togeter with operation + (addition) and scaler moltiplication s.t

1) + is commutative i.e $v + w = w + v$ for all v, w in V

2) + is associative, i.e $(v + w) + z = v + (w + z)$ for all v, w, z in V

3) There is an element in V, which we will denote as 0_V s.t $v + 0_V = 0_V + v = v$ for all v in V

4) Every v in V has an additive inverse, which we will denote $(-v)$ s.t $v + (-v) = (-v) + v = 0_V$

5) $1 \cdot v = v$ for all v in V

6) $(a \cdot b) v = a \cdot (b \cdot v)$ for all a, b in F and v in V

7) $a(v + w) = av + aw$ for all a in F and v, w in V

8) $(a + b) v = av + bv$ for all a, b in F, v in V

Ex: R^n over R

$$(x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n) = (x_1 + y_1, \dots, x_n + y_n)$$
$$\underset{v}{\quad} \quad \underset{w}{\quad} \quad \underset{v+w}{\quad}$$

$$\lambda(x_1, x_2 \dots x_n) = (\lambda x_1, \lambda x_2 \dots \lambda x_n)$$

Ex C^n over C

Ex C^2 over R

Is it really different from C^2 over C ?

yes   it has dimension 4 over R, a basis is:

$(1,0)$ $(i,0)$ $(0,1)$ $(0,i)$

Ex $P(F)$: the set of all polynomials with coefficient in F
with usual $+$, and multiplication by a constant, over F

Check   1) $P(x) + q(x) = q(x) + P(x)$

2) $\big(P(x) + q(x)\big) + r(x) = P(x) + \big(q(x) + r(x)\big)$

3) 0_v is the constant polynomial equal to 0

4) Given $P(x) = a_0 + a_1 x + \dots + a_n x^n$
$$P(x) + \big(-a_0 + (-a_1)x + \dots + (-a_n)x^n\big) = 0_v$$

5) $1 \cdot P(x) = P(x)$

6) $(ab) P(x) = a(b P(x))$

7) $a(P(x) + q(x)) = a P(x) + a q(x)$

8) $(a+b) P(x) = a P(x) + b P(x)$

Ex R^∞ : the set of all real sequences   with

$\{x_n\} + \{y_n\} = \{x_n + y_n\}$

$\lambda \{x_n\} = \{\lambda x_n\}$ over R

Ex   Given $S \subseteq R$, R^S : the set of all functions
$f : S \to R$ with $(f+g)(x) = f(x) + g(x)$
                $(\lambda f)(x) = \lambda \cdot f(x)$

over R

Ex $M_{n \times m}[F]$ the
set of all $n \times m$ matrices   with
coefficients in F   with usual   + and
scaler multiplication ; over   F
(Hw problem)

Some "algebra" properties of Vector Spaces:

Th: cancellation law for vector addition: $v + z = w + z \Rightarrow v = w$

for all v, w, z in V.

Proof: assume $v + z = w + z$, then

$v = v + 0 = v + (z + (-z)) = v + z + (-z) = w + z + (-z) = w$

Th: 0_v is unique, $0_F \cdot v = 0_v$ for all v in V, $a \cdot 0_v = 0_v$

for all a in F

Proof: suppose x in V has the property that

$x + v = v + x = v$ for all v in V then

$x + 0_v = 0_v$ (x additive identity) but also

$x + 0_v = x$ (0_v additive identity) so $x = 0_v$

$0_F \cdot v = \underline{0_F v + 0_v} = (0_F + 0_F) v + 0_v = 0_F v + \underline{0_F v + 0_v}$

by using cancellation property $0_v = 0_F v + 0_v = 0_F v$

$a \cdot 0_v = \underline{a 0_v + 0_v} = a(0_v + 0_v) + 0_v = a \cdot 0_v + \underline{a \cdot 0_v + 0_v}$

so again by cancellation law $0_v = a \cdot 0_v + 0_v = a \cdot 0_v$

Th: Additive inverse is unique

Proof Suppose $x + (-x) = x + y = 0_V$ then by cancellation law $-x = y$

Th: $(-1) \cdot v = -v$ for all v in V

$v + (-1)v = 1 \cdot v + (-1)v = (1-1) \cdot v = 0_v$

So $(-1) \cdot v$ is the additive inverse of v