Math 340 Winter 2021 final exam

Instructions

- You are allowed to consult the textbook and your notes.
- You are not allowed to discuss the exam with anybody, but of course, if anything is not clear you can email me your questions.
- Do not post questions or look for answers to any problem on the Web.
- Turn in your answer to Gradescope by Tuesday 3/16 11:59 pm, no late submissions will be accepted.

1. Let $M_{3\times 3}(\mathbb{R})$ be the vector space of 3x3 real matrices and let

$$U = \{M \in M_{3\times 3}(\mathbb{R}) : \text{ the sum of the diagonal elements of M is 0}\}$$

   (a) Show that U is a subspace of $M_{3\times 3}(\mathbb{R})$.

   (b) Find a basis for U. You need to justify (or it must be clear from your work) that your list of vectors is indeed a basis for U.

2. Let V be a vector space (not necessarily finite dimensional) and let S be a subset of V (not necessarily finite). Given a linear transformation $T : V \to V$, define the image of S under T to be the set

$$T(S) = \{w \in V : \exists v \in S,\, w = T(v)\}$$

   (a) Prove that if $T(S)$ is linearly independent, then S is linearly independent.

   (b) Is it true that if S is linearly independent then $T(S)$ must be linearly independent ? Justify your answer.

3. Suppose V is a vector space over F and $T : V \to F$ is a linear transformation. Suppose $v \in V$ and $T(v) \neq 0$. Prove that

$$V = ker(T) \oplus \operatorname{span}(v)$$

   Recall : $\operatorname{span}(v) = \{kv : k \in F\}$.

4. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation. Suppose v_1, v_2, v_3 are nonzero vectors in $\mathbb{R}^3$ such that:

$$(T - 2I)v_1 = 0,\ (T - 2I)v_2 = v_1,\ (T - 2I)v_3 = v_2$$

   Prove:

   (a) $\lambda = 2$ is the only eigenvalue for T.

   (b) T is not diagonalizable.

(c) Find a basis B for $\mathbb{R}^3$ such that $T_B^B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

5. Let V be a finite dimensional inner product space, and let U, W be subspaces of V. Prove that:

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

6. Let U=$span((1,1,0),(1,1,1))$. Find a vector $u \in U$ such that $||u - (1,0,0)||$ is as small as possible. Here $||\quad||$ is the usual norm in $\mathbb{R}^3$.