Math 318 Homework 9

(1) Using the command svd(A) in Julia you can compute the SVD of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \end{bmatrix}.$$

```
julia> using LinearAlgebra

julia> A = [1 1 0 0 0; 0 1 1 0 1; 1 2 1 0 1]
3Array{Int64,2}:
 1  1  0  0  0
 0  1  1  0  1
 1  2  1  0  1

julia> svd(A)
SVD{Float64,Float64,Array{Float64,2}}
U factor:
3Array{Float64,2}:
 -0.339287   0.742665   -0.57735
 -0.473523  -0.665163   -0.57735
 -0.81281    0.0775016   0.57735

singular values:
3-element Array{Float64,1}:
 3.253087102270064
 1.1905563006612325
 1.812986607347358e-16

Vt factor:
3Array{Float64,2}:
 -0.354155  -0.749574  -0.395419  0.0  -0.395419
  0.688894   0.195291  -0.493603  0.0  -0.493603
  0.632456  -0.632456   0.316228  0.0   0.316228
```

Mark the singular vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{v}_1, \mathbf{v}_2 \ldots$ that you see in the above SVD. You will need to show the grader what you marked. You can use these labels in answering the following questions and do not need to write down the vectors in the about output explicitly.

(a) What is the rank of A? Why?
(b) Find an orthonormal basis for the rowspace of A and columnspace of A.
(c) Which singular vectors lie in the nullspace of $A^\top$?
(d) Are some of the above singular vectors in the nullspace of A? If yes, which ones? Do we get a basis for the nullspace of A?
(e) Why is the fourth column of $V^\top$ filled with zeros?
(f) Write down the rank one decomposition of A. How many rank one matrices are there in the decomposition?
(g) What are the dimensions of the unit sphere and hyperellipse in the domain and codomain of the linear transformation given by A such that the hyperellipse is the image of the sphere?
(h) What are the semiaxes of the hyperellipse?

(2) (a) Use the SVD to argue that for any square matrix A of size $n \times n$, $|\det(A)| = \sigma_1 \sigma_2 \cdots \sigma_n$.

(b) (7.2 # 4) Compute the SVD of the following matrix (please do this by hand and not using software).

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Draw the unit sphere in the domain and its image hyperellipse in the codomain and mark all the left and right singular vectors and $\sigma_i \mathbf{u}_i$.

(c) Are the following true or false? If true explain why. If false, give a counterexample.

(i) A and $A^\top$ can have different singular values.

(ii) Let $A \in \mathbb{R}^{n\times n}$ and $\operatorname{rank}(A) = n$. Then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if all singular values of A equal 1.

(iii) Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be matrices of equal rank. Then A_1 and A_2 have the same singular values if and only if there exist orthonormal matrices Q_1, Q_2 such that $A_1 = Q_1 A_2 Q_2$.

(3) (7.3 #4) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

(a) Compute the best fitting line L and best fitting plane P to the four columns of A. Express L and P as the span of vectors.

(b) Compute the rank one decomposition of A, and the approximations A_1 and A_2.

(c) Check that the columns of A_1 are the projections of the columns of A on L.

(d) Check that the third column of A_2 is the projection of the third column of A on P.

(e) Compute the 2-norm of $A - A_1$.

(f) Construct a rank one matrix B of your choice and the same size as A. Check that B is not closer to A than A_1.

(4) † In this exercise we will develop **Gram-Schmidt orthogonalization** for converting a basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.

Input: A basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathbb{R}^n$.
Output: An orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_n$ of $\mathbb{R}^n$.

In the jth step, the *Gram-Schmidt algorithm* will find a unit vector $\mathbf{q}_j$ in $\operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_j\}$ that is orthogonal to $\mathbf{q}_1, \ldots, \mathbf{q}_{j-1}$. This means that after n steps, it produces an orthonormal basis.

In the following exercises, we are going to run this algorithm on the input and argue that at each step the algorithm does what it is supposed to. You will need to remember how to project onto a subspace that has an orthonormal basis (HW 8) and how orthogonal projectors and their complements work (HW 5).

(a) In step 1, set $\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}$. Argue that $\mathbf{q}_1$ is a unit vector and $\operatorname{Span}\{\mathbf{q}_1\} = \operatorname{Span}\{\mathbf{a}_1\}$.

(b) In Step 2, we want to find a unit vector $\mathbf{q}_2$ orthogonal to $\mathbf{q}_1$ and in $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

(i) Find the matrix P_1 projecting to $(\operatorname{Span}\{\mathbf{q}_1\})^\perp$.

(ii) Compute $\mathbf{a}_2' = P_1 \mathbf{a}_2$

(iii) Set $\mathbf{q}_2 = \frac{\mathbf{a}'_2}{\|\mathbf{a}'_2\|}$. Argue that $\mathbf{q}_2$ is a unit vector orthogonal to $\mathbf{q}_1$ and in $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

(c) Write down the jth step of this algorithm based on what you have seen so far:

(i) What is the formula for P_{j-1}, the matrix projecting to $(\operatorname{Span}\{\mathbf{q}_1, \dots, \mathbf{q}_{j-1}\})^\perp$?

(ii) Compute $\mathbf{a}'_j = P_{j-1}\mathbf{a}_j$

(iii) Set $\mathbf{q}_j = \frac{\mathbf{a}'_j}{\|\mathbf{a}'_j\|}$. Argue that $\mathbf{q}_j$ is a unit vector orthogonal to $\operatorname{Span}\{\mathbf{q}_1, \dots, \mathbf{q}_{j-1}\}$ and in $\operatorname{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_j\}$.

(d) Use the above algorithm to orthonormalize the basis consisting of

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$$

(5) * Watch the following video and write **at most one** page explaining how PCA can be used for face recognition. `https://www.youtube.com/watch?v=8BTv-KZ2Bh8` Write down two specific mathematical facts that you really liked in this video.

This problem will be graded with just 0,5 or 10 points. So it is important to write something very clear to get 10 points.