# Math 318 Homework 7

(1) (**Distance Realization**) Use Shoenberg's theorem to decide if the following distances are realizable. In other words, are there four points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in some $\mathbb{R}^k$ such that the $(i, j)$ entry in the matrix below is the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$?

$$\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & \sqrt{10} & \sqrt{5} \\ 3 & \sqrt{10} & 0 & \sqrt{13} \\ 2 & \sqrt{5} & \sqrt{13} & 0 \end{bmatrix}$$

If the distances are realizable, find four points that realize them (look at the proof of Shoenberg's theorem to see how you find the points).

(2) (**Differentiation is a linear transformation**) We showed in class that the set of all univariate polynomials of degree at most $d$, $\mathbb{R}[x]_{\le d}$ is a vector space. Consider a general polynomial of degree $d$:

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0.$$

Recall that we can identify $f(x)$ with its vector of coefficients $\mathbf{f} = (a_0, a_1, a_2, \ldots, a_d) \in \mathbb{R}^{d+1}$.

(a) Compute the derivative $f'(x)$ and write down its vector of coefficients.

(b) Argue that the function $D_d : \mathbb{R}[x]_{\le d} \to \mathbb{R}[x]_{\le d-1}$ that sends $f(x) \mapsto f'(x)$ is a linear transformation.

(c) Compute the derivatives of the elements in the monomial basis of $\mathbb{R}[x]_{\le d}$.

(d) Using the previous calculation write down the matrix $M_d$ of the linear transformation $D_d$.

(e) What does the $M_5$ matrix look like?

(f) Using $M_5$ express the derivative of $5x^5 - 19x^3 + 24x - 3$ as an image of the linear transformation $D_5$.

(g) Recall that $\mathbb{R}[x]$ is the (infinite-dimensional) vector space consisting of all polynomials of all degrees. Given any $p \in \mathbb{R}[x]$, use **linear algebra** to argue that there exists a polynomial $q \in \mathbb{R}[x]$ such that

$$5q'' + 3q' = p$$

An answer that uses integration **does not count!**

(3) (**Polynomial Interpolation**)

(a) Is there cubic polynomial $g(x)$ for which

$$g(-2) = -3, \; g(0) = 1, \; g(1) = 0, \; g(3) = 22$$

Show all your work.

(b) Suppose $t_1, t_2, \ldots, t_m$ are $m$ points on the real line $\mathbb{R}$. Consider the function that evaluates a polynomial of degree $d$ at $t_1, t_2, \ldots, t_m$:

$$\text{eval} : \mathbb{R}[x]_{\le d} \to \mathbb{R}^m \quad \text{such that} \quad f(x) \mapsto (f(t_1), f(t_2), \ldots, f(t_m))$$

We saw in the lecture notes that we can write $\text{eval}(f(x)) = M\mathbf{f}$ where $M$ is a Vandermonde matrix and $\mathbf{f}$ is the coefficient vector of $f(x)$. Show that eval is a linear transformation, i.e.,

(i) if $f(x), g(x) \in \mathbb{R}[x]_{\le d}$ then $\text{eval}(f(x) + g(x)) = \text{eval}(f(x)) + \text{eval}(g(x))$.

(ii) if $f(x) \in \mathbb{R}[x]_{\le d}$ and $\gamma \in \mathbb{R}$ then $\text{eval}(\gamma f(x)) = \gamma \text{eval}(f(x))$

(c) Compute the $3 \times 3$ Vandermonde matrix from $t_1, t_2, t_3$ and show that its determinant is $(-1)^3(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$.

(4) (**Solving a univariate polynomial equation**)

(a) Find the roots of the following polynomial using its companion matrix:

$$p(x) = x^7 - 3x^5 + 100x^4 - 2x - 5.$$

Then double check your answer by computing all roots of $p$ in Julia as in the lecture notes. The output from Julia must be included in your homework.

(b) Write down the companion matrix $A_p$ of the general degree 5 polynomial

$$p(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

and check that $p(x)$ is the characteristic polynomial of $A_p$.

(5) * **Equiangular lines in $\mathbb{R}^d$.** It is important in areas like coding theory to understand the largest number of lines through the origin in $\mathbb{R}^d$ such that the angle between any two of them is the same. A collection of lines through the origin in $\mathbb{R}^d$ such that the the angle between any two of them is the same is a set of *equiangular* lines. For example, the maximum number of lines in $\mathbb{R}^3$ such that the angle between any two of them is $90°$ is 3; take for example the 3 coordinate axes. However, if the common angle is different from $90°$ there can be more lines. The 6 diagonals of a regular icosahedron are equiangular. Google for the regular icosahedron if you have not seen it before.

In the following exercise we will argue that you cannot have more than 6 equiangular lines in $\mathbb{R}^3$ no matter what angle $\theta$ you choose.

Suppose we have a collection of $n$ equiangular lines in $\mathbb{R}^3$ and $\mathbf{v}_i$ is a unit vector in the direction of the $i$th line. It does not matter if you choose $\mathbf{v}_i$ or its negative, but choose one and call it $\mathbf{v}_i$.

(a) Argue that the condition of equiangularity means that if $i \neq j$ then $\mathbf{v}_i^\top \mathbf{v}_j = \cos\theta$ for a fixed angle $\theta$.

(b) You showed last week that the set of all symmetric $3 \times 3$ matrices form a vector space. Argue that the dimension of this vector space is 6.
**Hint:** What is a basis of the space of all $2 \times 2$ symmetric matrices? Then try $3 \times 3$.

Now consider the rank one PSD matrices $\mathbf{v}_i\mathbf{v}_i^\top$ of size $3 \times 3$.

(c) We will now argue that the rank one psd matrices $\mathbf{v}_i\mathbf{v}_i^\top$ of size $3 \times 3$, coming from the equiangular lines, are **linearly independent** (as matrices). This will give us the result since all these rank one psd matrices are in the 6 dimensional vector space of symmetric matrices, so there cannot be more than 6 of them. This will imply that there cannot be more than 6 lines.

(i) Suppose the **matrices** $\mathbf{v}_i\mathbf{v}_i^\top$ are **linearly dependent**. Then there are $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\sum a_i \mathbf{v}_i\mathbf{v}_i^\top = 0.$$

Multiply this expression on the left with $\mathbf{v}_j^\top$ and on the right with $\mathbf{v}_j$ and get that

$$0 = a_j + \sum_{i \neq j} a_i \cos^2\theta.$$

(ii) Find a matrix $M$ such that you can express the equations from (b) in the form $M\mathbf{a} = 0$.

(iii) Check that $M = (1 - \cos^2\theta)I_n + \cos^2\theta J_n$ where $J_n$ is the $n \times n$ matrix of all ones.

(iv) Argue that $I_n$ and $J_n$ are PSD.

(v) Argue that the coefficient $(1 - \cos^2\theta)$ is positive and hence $M$ is PSD.

(vi) Is $M$ positive definite? If so, what is $\mathbf{a}$ if $M\mathbf{a} = \mathbf{0}$?

(vii) Conclude that the psd matrices $\mathbf{v}_i\mathbf{v}_i^\top$ are linearly independent.

(d) Can you extend each step above to see that you cannot have more than $\binom{d+1}{2} = \frac{d(d+1)}{2}$ equiangular lines in $\mathbb{R}^d$?