Math 318 Homework 6

(1) **Symmetric Matrices** (6.4 # 7)

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

The following are eigenvalue/eigenvector pairs of A:

$$\lambda_1 = -3, \mathbf{u}_1 = (1,-2,-2)^\top, \quad \lambda_2 = 0, \mathbf{u}_2 = (2,2,-1)^\top, \quad \lambda_3 = 3, \mathbf{u}_3 = (2,-1,2)^\top$$

- (a) Find a set of orthonormal eigenvectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of A where $A\mathbf{q}_i = \lambda_i \mathbf{q}_i$.
- (b) Compute a orthonormal diagonalization of A.
- (c) Compute the coordinates of $(1,1,1)^\top$ in the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.
- (d) Compute the coordinates of $A(1,1,1)^\top$ in the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

(2) **Quadratic forms, PSD matrices**

- (a) Argue that for each n, the only orthonormal, symmetric, and positive definite matrix of size $n \times n$ is the identity.
- (b) (6.5 #12) For what values of d (if any) is the following matrix is positive definite? Show your work and explain your conclusion.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

- (c) Let $q(x,y) = x^2 - 4xy + 16y^2$.
  - (i) Write down the symmetric matrix Q such that $q(x,y) = \begin{pmatrix} x & y \end{pmatrix} Q \begin{pmatrix} x \\ y \end{pmatrix}$.
  - (ii) Do you expect $q(x,y)$ to be nonnegative for all $(x,y) \in \mathbb{R}^2$? Why? If yes, write it as a sum of squares by factoring Q as $BB^\top$.

(3) **PSD matrices**

- (a) Argue that the set of all $n \times n$ symmetric matrices forms a subspace of $\mathbb{R}^{n\times n}$.
- (b) Argue that for $n \times n$ matrices,
  - (i) if $A \geq 0$ and $B \geq 0$, then $A + B \geq 0$,
  - (ii) if $A \geq 0$ and $\lambda \geq 0$ then $\lambda A \geq 0$.
- (c) Does the set of all $n \times n$ PSD matrices form a subspace in the space of all $n \times n$ symmetric matrices? Explain your reasons.

(4) * **Equiangular lines in $\mathbb{R}^d$.**

It is important in areas like coding theory to understand the largest number of lines through the origin in $\mathbb{R}^d$ such that the angle between any two of them is the same. A collection of lines through the origin in $\mathbb{R}^d$ such that the the angle between any two of them is the same is a set of *equiangular* lines. For example, the maximum number of lines in $\mathbb{R}^3$ such that the angle between any two of them is 90° is 3; take for example the 3 coordinate axes. However, if the common angle is different from 90° there can be more lines. The 6 diagonals of a regular icosahedron are equiangular. Google for the regular icosahedron if you have not seen it before.

In the following exercise we will argue that you cannot have more than 6 equiangular lines in $\mathbb{R}^3$ no matter what angle θ you choose.

Suppose we have a collection of n equiangular lines in $\mathbb{R}^3$ and $\mathbf{v}_i$ is a unit vector in the direction of the ith line. It does not matter if you choose $\mathbf{v}_i$ or its negative, but choose one and call it $\mathbf{v}_i$.

(a) Argue that the condition of equiangularity means that if $i \neq j$ then $\mathbf{v}_i^\top \mathbf{v}_j = \cos\theta$ for a fixed angle θ.

(b) You showed in problem 3 that the set of all symmetric 3×3 matrices form a vector space. Argue that the dimension of this vector space is 6.
Hint: What is a basis of the space of all 2×2 symmetric matrices? Then try 3×3.

Now consider the rank one PSD matrices $\mathbf{v}_i\mathbf{v}_i^\top$ of size 3×3.

(c) We will now argue that the rank one psd matrices $\mathbf{v}_i\mathbf{v}_i^\top$ of size 3×3, coming from the equiangular lines, are **linearly independent** (as matrices). This will give us the result since all these rank one psd matrices are in the 6 dimensional vector space of symmetric matrices, so there cannot be more than 6 of them. This will imply that there cannot be more than 6 lines.

(i) Suppose the **matrices** $\mathbf{v}_i\mathbf{v}_i^\top$ are **linearly dependent**. Then there are $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\sum a_i \mathbf{v}_i \mathbf{v}_i^\top = 0.$$

Multiply this expression on the left with $\mathbf{v}_j^\top$ and on the right with $\mathbf{v}_j$ and get that

$$0 = a_j + \sum_{i \neq j} a_i \cos^2\theta.$$

(ii) Find a matrix M such that you can express the equations from (i) in the form $M\mathbf{a} = 0$.

(iii) Check that $M = (1 - \cos^2\theta)I_n + \cos^2\theta J_n$ where J_n is the $n \times n$ matrix of all ones.

(iv) Argue that I_n and J_n are PSD.

(v) Argue that the coefficient $(1 - \cos^2\theta)$ is positive and hence M is PSD.

(vi) Is M positive definite? If so, what is $\mathbf{a}$ if $M\mathbf{a} = \mathbf{0}$?

(vii) Conclude that the psd matrices $\mathbf{v}_i\mathbf{v}_i^\top$ are linearly independent.

(d) Can you extend each step above to see that you cannot have more than $\binom{d+1}{2} = \frac{d(d+1)}{2} = 1 + 2 + \cdots + d$ equiangular lines in $\mathbb{R}^d$?