Math 318 Homework 5

(1) Find the best fit line through the points $(0,0),(1,3),(3,4),(5,7)$. You should show and explain all steps of the calculation and state explicitly the formulae you are using. Do not use a software package to find the best fit line.
  (a) Draw the points and best fit line.
  (b) What is the value of $e_1^2+e_2^2+e_3^2+e_4^2$ that the line minimizes?

(2) (6.4 # 7) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

The following are eigenvalue/eigenvector pairs of A:

$$\lambda_1 = -3, \mathbf{u}_1 = (1,-2,-2)^\top, \quad \lambda_2 = 0, \mathbf{u}_2 = (2,2,-1)^\top, \quad \lambda_3 = 3, \mathbf{u}_3 = (2,-1,2)^\top$$

  (a) Find a set of orthonormal eigenvectors $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$ of A where $A\mathbf{q}_i = \lambda_i\mathbf{q}_i$.
  (b) Compute a orthonormal diagonalization of A.
  (c) Compute the coordinates of $(1,1,1)^\top$ in the basis $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$.
  (d) Compute the coordinates of $A(1,1,1)^\top$ in the basis $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$.

(3) † (**(Oblique) Projectors**) A square matrix $P \in \mathbb{R}^{n\times n}$ is a *projector* if $P^2 = P$. In Homework 2 you showed that such a P has at most two eigenvalues, 0 and 1, and that $E_0 = \mathrm{Null}(P)$ and $E_1 = \mathrm{Col}(P)$.

We'll see that such matrices model *oblique* projections as opposed to the *orthogonal* projections we saw in class. If $P \in \mathbb{R}^{n\times n}$ is a projector and $\mathbf{v} \in \mathbb{R}^n$, call $P\mathbf{v}$ the *shadow* of $\mathbf{v}$ under P. Note that shadows live in $\mathrm{Col}(P)$.

  (a) Consider the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

    (i) Show that $P^2 = P$, i.e., P is a projector.
    (ii) Compute and draw its column space and nullspace in $\mathbb{R}^2$.
    (iii) For $\mathbf{x} = (5,2)^\top$, compute $P\mathbf{x}$ and $\mathbf{x} - P\mathbf{x}$ and locate them in your picture. Notice that $\mathbf{x} = P\mathbf{x} + (\mathbf{x} - P\mathbf{x})$.

  (b) Suppose P is a projector. (*Use the above example to check each of the steps in this question. You do not need to show any work on the example.*)
    (i) Check that if $\mathbf{v} \in \mathrm{Col}(P)$, then $P\mathbf{v} = \mathbf{v}$.
    (ii) Check that for all $\mathbf{v}$, then $\mathbf{v} - P\mathbf{v} \in \mathrm{Null}(P)$.
    Since $\mathbf{v} = P\mathbf{v} + (\mathbf{v} - P\mathbf{v})$, *we say that* $P\mathbf{v}$ *is the oblique projection or shadow of* $\mathbf{v}$ *into* $\mathrm{Col}(P)$ *along* $\mathrm{Null}(P)$. *Check in example above.*
    (iii) Show that $I - P$ is also a projector. *This is called the complementary projector to* P. *Compute it in the example to keep the example going.*
    (iv) Argue that $I - P$ obliquely projects onto $\mathrm{Null}(P)$, i.e., you need to show that $\mathrm{Col}(I-P) = \mathrm{Null}(P)$.
    (v) Now using that $P = I - (I - P)$ or otherwise, argue that $\mathrm{Col}(P) = \mathrm{Null}(I-P)$.
    (vi) Use the previous part to argue that $\mathrm{Col}(P) \cap \mathrm{Null}(P) = \{\mathbf{0}\}$.
    So far we have that P *projects onto* $\mathrm{Col}(P)$ *and* $I - P$ *projects onto* $\mathrm{Null}(P)$ *and that these spaces are complementary in the sense that they only share the origin. Both projections are oblique, not orthogonal.*

(vii) Show that you can write any vector $\mathbf{v}$ uniquely as the sum of vectors in $\mathrm{Col}(P) = E_1$ and $\mathrm{Null}(P) = E_0$.

(4) † (**Orthogonal Projectors**) $P \in \mathbb{R}^{n\times n}$ is called an orthogonal projector if $P^2 = P$ and $P = P^\top$, i.e., P is a symmetric projector. Since orthogonal projectors are projectors, all the results from the previous problem still apply, so use them, but some special things happen now because of symmetry.

(a) Consider $P = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$.

(i) Show that P is an orthogonal projector. i.e., $P^2 = P$ and $P = P^\top$.

(ii) What space does it project onto?

(iii) Draw a picture and mark E_0 and E_1.

(iv) For $\mathbf{v} = (10, 20)^\top$, find $P\mathbf{v}$ and $\mathbf{v} - P\mathbf{v}$ and mark them in your picture.

(b) Suppose P is an orthogonal projector.

(i) Argue that $\mathrm{Col}(P)$ and $\mathrm{Null}(P)$ are orthogonal complements. *This is why we call P an orthogonal projector.*

(ii) Check that the projector $P = A(A^\top A)^{-1}A^\top$ from Chapter 4.3 in the lecture notes, that projects to $\mathrm{Col}(A)$, is an orthogonal projector. (**Hint**: You may use the fact that for an invertible matrix B, $(B^{-1})^\top = (B^\top)^{-1}$)

(iii) Argue that $\mathrm{Col}(A) = \mathrm{Col}(A(A^\top A)^{-1}A^\top)$. (**Hint**: Recall where a projector P projects to from Problem (4) and recall from lecture notes where $A(A^\top A)^{-1}A^\top$ projects to.)

(c) Let $\mathbf{u} \in \mathbb{R}^n$ be a vector of length 1 (unit norm) and consider $H = I - \mathbf{u}\mathbf{u}^\top$.

(i) Argue that H is an orthogonal projector.

(ii) Express the space that H projects onto, in terms of $\mathbf{u}$.

(iii) Find all the eigenspaces of H and their dimensions.

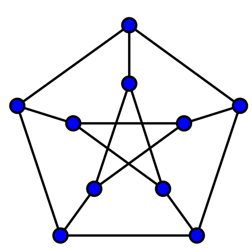
(iv) What does the linear transformation $R = I - 2\mathbf{u}\mathbf{u}^\top$ do? Explain with reasons.

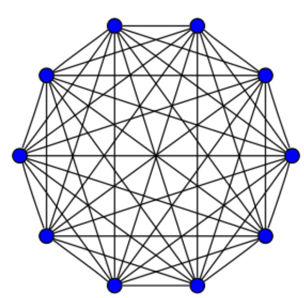
(5) * **The Petersen Puzzle**

Before you attempt this problem, look at the article about Ringel's conjecture in Quanta Magazine: https://www.quantamagazine.org/mathematicians-prove-ringels-graph-theory-conjecture-20200219/. The animations there illustrate the notion of tiling needed in this problem.

Below you see two important graphs:

- On the left is the *Petersen graph* with 10 nodes and 15 edges.
- On the right is the *complete graph* on 10 vertices called K_{10} with 10 nodes and 45 edges (i.e., all possible edges among 10 nodes).





Each node in K_{10} has 9 edges incident to it while each node in the Petersen graph has 3 edges incident to it. So it is plausible that K_{10} can be covered perfectly (the technical word is *tiled*) by 3 Petersen graphs. This means that you can lay down three Petersens on K_{10} so that vertices go to vertices and each edge of K_{10} lies under an edge of exactly one of the three Petersens. In the exercise below we will use several things we have learned so far to argue that it is NOT possible to cover K_{10} with 3 Petersens.

Fact: The adjacency matrix of the Petersen graph has eigenvalue 1 *with multiplicity* 5*. It does not have* -3 *as an eigenvalue.*

(a) If J_{10} is the 10×10 matrix with all entries equal to 1 and I_{10} is the 10×10 identity matrix, then argue that the adjacency matrix of K_{10} is $J_{10} - I_{10}$.

(b) If there were three Petersen graphs called P, Q, R that cover K_{10}, and their adjacency matrices are A_P, A_Q, A_R, then argue that

$$A_P + A_R + A_Q = J_{10} - I_{10}.$$

(c) Argue that the matrix $A_P - I_{10}$ has a 5-dimensional nullspace.
(**Hint**: Look at the given multiplicity of 1 as an eigenvalue of A_P. What property of A_P can you use to understand the dimension of $A_P - I_{10}$?)

(d) Argue that the nullspace of $A_P - I_{10}$ is in the orthogonal complement of $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{10}$. **Hint**: Where must $\mathbf{1}$ lie for this to be true?

(e) What is the dimension of the orthogonal complement of $\mathbf{1}$?

(f) The above results are also true for $A_Q - I_{10}$ since Q is also a Petersen graph. Therefore, argue using dimensions of the subspaces you have looked at, that
   (i) there is a non-zero vector $\mathbf{w}$ in the intersection of nullspace$(A_P - I_{10})$ and nullspace$(A_Q - I_{10})$, and
   (ii) $\mathbf{1}^\top \mathbf{w} = 0$.

   Hint for (i): Can there be two subspaces in $\mathbb{R}^n$ of dimensions a and b that only intersect at the origin if $a + b > n$? Try some small examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ to gain some intuition.

(g) Now compute $A_R\mathbf{w}$ using the expression $A_R = (J_{10} - I_{10} - A_P - A_Q)$ from (b) and observe that -3 is an eigenvalue of A_R.

(h) What can you conclude?