Math 318 Homework 4

(1) (4.1 #3) Construct a matrix with the required property or explain why you cannot:

(a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) Row space contains $\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}$ and $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution and $A^\intercal \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(d) Every row is orthogonal to every column but the matrix is not the zero matrix.

(e) The sum of columns is the zero vector and the sum of rows is the vector of all ones.

(2) The following three parts are unrelated.

(a) Let

$$S = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} \right\}$$

Find a basis for $S^\perp$.

(b) Let V be the following 3-dimensional plane in $\mathbb{R}^4$:

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\}$$

Find a basis for $V^\perp$.

(c) If W is a subspace of $\mathbb{R}^3$ containing only the origin, what is $W^\perp$?

(3) (4.2 #30)

(a) Find the projection matrix P_C onto the column space of

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}.$$

Look carefully at this matrix before you start. Remember that $A^\intercal A$ is invertible if and only if the columns of A are linearly independent.

(b) Find the projection matrix P_R onto the row space of A.

(4) (4.1, #6, #7) The following system $A\mathbf{x} = \mathbf{b}$ has no solution. You can check if you like, or just believe me.

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

(a) Find a vector $\mathbf{y} \in \mathbb{R}^3$ such that $\mathbf{y}^\intercal A = 0$ and $\mathbf{y}^\intercal \mathbf{b} \neq 0$.

(b) If you could produce such a $\mathbf{y}$ can you convince your boss that $A\mathbf{x} = \mathbf{b}$ has no solution without solving the system?
Hint: Can you derive a contradiction if there was some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ and a $\mathbf{y}$ such that $\mathbf{y}^\top A = 0$ and $\mathbf{y}^\top \mathbf{b} \neq 0$.

(c) Which subspace associated to A did $\mathbf{y}$ come from?

(d) Is it true that whenever $A\mathbf{x} = \mathbf{b}$ has no solution there will be a $\mathbf{y}$ such that $\mathbf{y}^\top A = 0$ and $\mathbf{y}^\top \mathbf{b} \neq 0$? Explain.
Hint: If $A\mathbf{x} = \mathbf{b}$ has no solution how does the echelon form $\left[B \,|\, \mathbf{b}'\right]$ of the augmented matrix $\left[A \,|\, \mathbf{b}\right]$ look? What is the relationship between the solutions of $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}'$? If $B\mathbf{x} = \mathbf{b}'$ has no solution, can you easily produce a $\mathbf{z}$ such that $\mathbf{z}^\top B = 0$ and $\mathbf{z}^\top \mathbf{b}' \neq 0$? Can you write an equation that expresses $\left[B \,|\, \mathbf{b}'\right]$ in terms of $\left[A \,|\, \mathbf{b}\right]$ and use that to produce the $\mathbf{y}$ we are looking for?

(5) **A shop with no small change***

This problem has many parts and a lot of commentary. For clarity, here is the color coding we will use:
(i) problem statements, (ii) math formulations, (iii) commentary/philosophy,
(iv) running example, (v) ♠ action item for you (needs to be written up)

Suppose you own an art supply shop that sells n different items (n is very large), and suppose m children have placed orders for the start of school (m is much smaller than n). Suddenly all coins of value less than \$1 go out of circulation, and you now need to round the prices of your items up or down. How can you round the prices so that the total price of each order is not affected too much? We will show that the following is possible using linear algebra.

Theorem 1. *Suppose at most t items of each type has been ordered in total, and no order asks for more than one item of each type. Then it is possible to round the prices so that the total price of each order changes by no more than t dollars.*

Mathematical formulation:
- Suppose the price of item j is c_j. Note that we can assume $0 < c_j < 1$ for all j since only the rounding matters.
- Since each order contains only one of each item, we can represent an order S_i as a subset of $\{1, 2, 3, \ldots, n\}$. For example, if $S_1 = \{2, 6, 9\}$ then child 1 has ordered one of item 2, item 6 and item 9.
- We are also told that for each j, item j is in no more that t sets among $S_1, \ldots, S_m$. (*Note the role of t in the theorem. Don't forget what t is!*)
- The theorem says that we can find numbers $z_1, \ldots, z_n \in \{0, 1\}$ (the rounded up/down prices for the n items) so that each order changes in price by at most t dollars. The change in price of order S_i is $|\sum_{j \in S_i} (c_j - z_j)|$. So we'll get

(1) $$\Big|\sum_{j \in S_i} (c_j - z_j)\Big| \leq t \quad \text{for all} \quad i = 1, 2, \ldots, m$$

Running example: Suppose you carry $n = 7$ items in your shop, $m = 3$ children place orders, and no more than $t = 2$ items of each type are ordered in total. The three orders could be:

$$S_1 = \{1, 2, 3, 5, 7\}, S_2 = \{1, 2, 6, 7\}, S_3 = \{3, 4, 5, 6\}.$$

Further suppose the costs of the 7 items are

$$c_1 = \frac{1}{2},\ c_2 = \frac{1}{4},\ c_3 = \frac{1}{4},\ c_4 = \frac{3}{4},\ c_5 = \frac{1}{2},\ c_6 = \frac{1}{4},\ c_7 = \frac{3}{4}.$$

By the theorem, we will be able to round (up or down) the 7 prices to $z_1, \ldots, z_7$ each of which will be 0 or 1, so that each order changes by at most \$2 ($t = 2$).

(a) ♠ Suppose each order contained at most s items. Argue that we can easily round prices so that each order changes by at most s dollars.
In our example, $s = 5$.

So the interesting part about this theorem is that you can do much better if s is large, i.e., each order has lots of items, but t is small, i.e., the total number of erasers (or easels, or whatever) ordered is small.
In our running example, it is easy to round prices so that no order changes by more than \$5 in price, but the cool thing is that we can get the change in price to be at most \$2.
Note that the theorem doesn't care what the original prices c_j are.

The algorithm: The way to round prices is via the following *iterative algorithm*. This means we will repeat a procedure over and over again until we get what we want. Comments are in italics.

Initialize: For each item j, let x_j be a **floating variable** that is initially set to c_j.

The following iterative method will move each floating x_j to 0 or 1 which then becomes the value of z_j (the rounded price). Once this happens, x_j is permanently set to z_j and we say x_j becomes **fixed**. *In each step of our procedure, at least one floating variable will become fixed.*

Call S_i **dangerous** if it has more than t indices j for which x_j is still floating; the others sets are **safe**. In our running example, all sets are dangerous at the start and all variables are floating. We will always maintain the following equality:

(2) $$\sum_{j \in S_i} x_j = \sum_{j \in S_i} c_j \quad \text{for all dangerous sets } S_i$$

At the start, all variables are floating and the above equation is true since $x_j = c_j$ for all j.

(i) Write down the equations (2) for all the currently dangerous sets. Think of this as a system of linear equations with the floating variables as unknowns and the fixed variables as constants. Find a solution of this system where at least one of the floating variables becomes 0 or 1. *We will argue later that this is always possible.*

(ii) If x_j gets set to 0 or 1 then set z_j to the value of x_j. Declare x_j fixed. *Several x_j's can get fixed at the same time.*

(iii) If there are no more dangerous sets, then stop. Otherwise, write down the new system of linear equations (2) with all the fixed x_j's turned into z_j's and thought of as constants. *Do not remove anything from the old system; simply replace x_j by it's fixed value z_j in each equation. A fixed x_j is no longer a variable – it has a value — and the number of x_j variables have decreased from the old system to the new system.* Go back to step (i).

(b) ♠ Run the above algorithm on our example and check that the theorem is true.

In the rest of this problem we argue that the algorithm always produces prices as stated in the theorem.

(c) ♠ Argue that at the start, the linear system in (i) is feasible.

(d) You can say a bit more. Suppose F is the set of indices of the floating variables in the system and $|F|$ denotes the cardinality of F, i.e., the number of elements in F. In our example, at the start, $F = \{1, 2, 3, 4, 5, 6, 7\}$ and so $|F| = 7$.

♠ Argue that the system has a solution strictly inside the unit cube $[0,1]^{|F|}$. **Hint**: this is a one line answer, just think about what the values of x_j are at the start.

By $[0,1]^k$ *we mean a cube with side lengths* 1 *and with opposite corners* $(0,0,\dots,0)$ *and* $(1,1,\dots,1)$ *in* $\mathbb{R}^k$*. See Figure 1 to see examples and what it means to be strictly inside the cube* $[0,1]^{|F|}$.

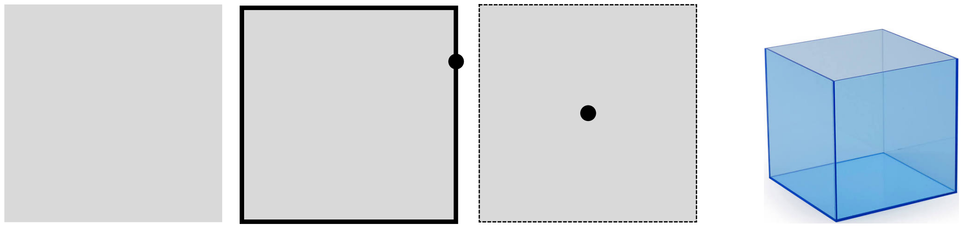

FIGURE 1. *On the left you see the square* $[0,1]^2$ *which is the unit cube in dimension* 2*. Next you see the boundary of the square in thick lines and a point on the boundary. Then you see the square without its boundary and a point strictly inside the square (i.e., not on the boundary). At the end is the unit cube* $[0,1]^3$*, which is the usual cube. Its boundary consists of the* 6 *squares that form the outside of the cube. A point is strictly inside the cube if it does not lie on any of these* 6 *squares.*

(e) Suppose now you are in some iteration of the algorithm.
   (i) ♠ Argue that there are fewer dangerous sets than floating variables in step (i).
   In our example, at the start, we have 3 dangerous sets and 7 floating variables at the start since all variables are floating at the start, and indeed $3 < 7$.
   This is the trickiest part of the question, so let's break it down. Suppose there are f floating variables and d dangerous sets.
      (A) ♠ Each floating variable can be in at most t dangerous sets. So argue that if we sum up the number of floating variables in each dangerous set, we get at most ft.
      (B) ♠ Each dangerous set contains at least $t+1$ floating variables. So if we again sum up the number of floating variables in each dangerous set, we get at least $d(t+1)$.
      (C) ♠ Write the inequality that relates ft and $d(t+1)$ and conclude that $d < f$.
   (ii) ♠ Argue that the linear system obtained by taking all the equations (2) as you vary over dangerous sets S_i has a solution space of dimension at least one, i.e., contains a line l. **Hint**: Use the previous part and think about when linear systems of equations always have a solution.
   (iii) ♠ Now argue that there is a solution to the linear system that lies on the boundary of the cube $[0,1]^{|F|}$. **Hint**: use the line l to find such a point $\mathbf{y}$.
   (iv) ♠ Use the coordinates of $\mathbf{y}$ as the values of the floating variables. Argue that at least one floating variable will get set to 0 or 1.

(f) We keep going through iterations of the algorithm, each time setting up equations (2) and fixing some floating variables.
   (i) ♠ Argue that after finitely many iterations, there will be no more dangerous sets and the algorithm will stop.
   (ii) ♠ How many iterations will be needed at worst?

(g) To finish we need to argue that when there are no more dangerous sets, we will have the inequality (1).

   (i) ♠ Consider an S_i. Argue that it satisfies (2) and has at most t floating variables.

   (ii) ♠ If there are at most t floating variables in each set S_i, can you conclude that the we have (1)? **Hint**: Do you see a way to round the values of the remaining floating variables to 0 or 1 (at most t of them) so that $|\sum_{j \in S_i}(c_j - z_j)| \le t$?