Math 318 Homework 3

(1) (6.2 #9) Suppose a sequence $\{G_k\}$ is defined as $G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$.

(a) Find the matrix A such that

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(b) Find the eigenvalues and eigenvectors of A.

(c) Find the limit as $n \to \infty$ of the matrices A^n.

(d) If $G_0 = 0$ and $G_1 = 1$ show that G_n approaches $\frac{2}{3}$ as $n \to \infty$.

(2) (10.3 #6 and #9) Suppose A is a $n \times n$ non-negative Markov matrix.

(a) Argue that if $\mathbf{u} \in \mathbb{R}^n$ is a nonnegative vector ($\mathbf{u} \neq \mathbf{0}$) whose entries add to α, then $A\mathbf{u}$ is a nonnegative vector whose entries add to α.
Hint: Use the all ones vector $\mathbf{1}$ from the lecture notes and make an argument using matrix-vector multiplication without writing out A or $\mathbf{u}$.

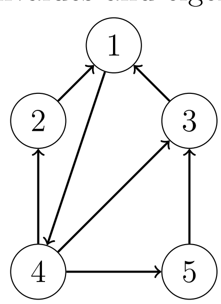
(b) Suppose $A\mathbf{v} = \lambda\mathbf{v}$ where $\lambda \neq 1$ and $\mathbf{v}$ is a non-zero vector whose entries add up to α. Then what must α be? Illustrate on a 2×2 Markov matrix of your choice.

(c) (10.3 #11) Complete the following to a positive Markov matrix

$$\begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & - \\ - & - & - \end{bmatrix}$$

so that $(1, 1, 1)^\intercal$ is the dominant eigenvector. What is the dominant eigenvalue? Explain everything you did, and your logic, in words.

(3) Compute the page rank of the 5 webpages you see in the following directed graph. An arrow from i to j indicates that page i links to page j. Use the rule that when you are at a webpage you will teleport with probability $\frac{1}{2}$ and follow a link with probability $\frac{1}{2}$. Assume that all links from a page have equal probability of being followed. You can also assume that you will teleport to any of the vertices in the network with equal probability as we assumed in class. (You will want to use a software package to compute eigenvalues and eigenvectors.)



(4) (4.1, #4, #30) Recall from class that for any vector $\mathbf{x}$, $A\mathbf{x}$ is a linear combination of the columns of A and for any vector $\mathbf{y}$, $\mathbf{y}^\top A$ is a linear combination of the rows of A. Check this if you are not convinced.

   (a) Let A and B be two matrices such that AB is defined. What is the relationship between
      (i) column space of AB and column space of A?
      (ii) row space of AB and rowspace of B?
      (iii) Using (i) and (ii) argue that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.
      (iv) How can you use (iii) to see that if the columns of a matrix $A \in \mathbb{R}^{n\times k}$ are not linearly independent, then $A^\top A$ cannot be inverted? In class we showed (or will see) the converse statement, namely, *if the columns of A are linearly independent then $A^\top A$ is invertible.*

   (b) Now suppose $AB = 0$. What is the relationship between
      (i) $\text{Null}(A)$ and $\text{Col}(B)$?
      (ii) $(\text{Col}(B))^\perp$ and $\text{Row}(A)$?

   (c) If $AB = 0$, can A and B be 3×3 matrices of rank 2?

   (d) Suppose $A \in \mathbb{R}^{3\times 4}$ and $B \in \mathbb{R}^{4\times 5}$ and $AB = 0$. Argue that $\text{rank}(A) + \text{rank}(B) \leq 4$.

(5) ***How to limit clubs in your community.** Suppose there are n people in your community and the community leaders are feeling overwhelmed by the number of different clubs that are getting formed (for which they need to fund cookies and drinks each week). The leaders devise the following rules to limit the number of clubs that can be formed:

   (a) *Each club has to have an odd number of members.*
   (b) *Every two clubs must have an even number of members in common.*

   Let's use linear algebra to argue that no more than n clubs can be formed under these rules. Make a small example that you can keep using as you do the various parts of this question. Always do small examples! Examples are very enlightening. Also, in these star problems, feel free to use sentences to explain your logic.

   (a) Let's call the members of the community $1, 2, \ldots, n$ and the clubs $C_1, \ldots, C_m$. Form the $m \times n$ matrix with rows indexed by clubs and columns by people as follows:
$$a_{ij} = \begin{cases} 1 \text{ if person } j \text{ is in club } C_i \\ 0 \text{ otherwise} \end{cases}$$
   Write an inequality that relates $\text{rank}(A)$ and n.

   (b) Now compute $AA^\top$ which is an $m \times m$ matrix. Argue that the (i,k) entry of $AA^\top$ counts the number of people common to both club C_i and club C_k. In particular, the (i,i) entry counts the number of people in C_i.

   (c) Next we replace all the odd numbers you see in $AA^\top$ with 1 and all the even numbers with 0. (In mathematical language we are working in the field F_2 with two elements 0 and 1 where $1 + 1 = 0$. Or equivalently, we are computing mod 2. Don't worry if haven't seen this before.) After you have made these replacements, what is $AA^\top$ given the rules on clubs? What is $\text{rank}(AA^\top)$?

   (d) Using the result of probem 5 (a) iii), and the previous step, argue that $m \leq n$. In other words your community of n people cannot form more than n clubs under the rules.