Math 318 Homework 1

The following applies to all homework sets in this class:
In writing solutions to all homework, please show your work, including all computations. Also, feel free to explain your reasoning in words, as needed. **Partial credit cannot be given without seeing your complete thought process and reasoning.**

Many of the questions in homework are inspired by problems in *Introduction to Linear Algebra, Fifth Edition* by Gilbert Strang. In these cases, the problem number from Strang's book is written at the start of the question (e.g. Problem (3) below is from Chapter 6.1 Problem #16 and Chapter 6.2 Problem #20, in Strang's book).

Starting in the next homework, you can use software for all computations you learned in Math 308 such as determinants, eigenvalues etc, but on THIS homework please do all computations by hand.

(1) **Finding Eigenspaces**

Compute all eigenvalues of the following matrix and a basis for each eigenspace:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Answer the following questions using your computations:

(a) What are the eigenvalues and eigenvectors of A^3 and $A + 10I$?
(b) What are the eigenvalues and eigenvectors of $A^\top$? In general, how are the eigenvalues of A related to the eigenvalues of $A^\top$? **Explain**.
(c) Is A diagonalizable? If yes, write its diagonalization and compute A^3.
(d) From your eigenvalue computation, decide if A is invertible. If yes, what are the eigenvalues of A^{-1}?
(e) Can you tell definitively from the eigenvalues of a square matrix A whether A is invertible? If yes, say how. If not, give an example or reason to justify your answer.

(2) **Invertibility vs. Diagonalizablilty**

In each of the following cases, find an example of a matrix that satisfies the given conditions or say why there can be no such matrix. You must explicitly show the diagonalization of the matrix you chose or explain why your matrix cannot be diagonalized by computing eigenvalues and eigenvectors. Small matrices will work in all cases — 2×2 or 3×3.

(a) a matrix that is invertible and diagonalizable
(b) a matrix that is invertible but not diagonalizable
(c) a matrix that is singular but diagonalizable
(d) a matrix that is singular and not diagonalizable
(e) What can you conclude about the relationship between invertibility and diagonalizability of a matrix?

(3) **Similarity of Matrices**

(6.2 #38) Recall that a matrix A is *similar* to a matrix C if there is an invertible matrix B such that $A = BCB^{-1}$. Also, we saw in class that similar matrices have the same eigenvalues. Suppose Λ is the diagonal matrix with the eigenvalues of A on its diagonal.

(a) Are A and Λ always similar? If yes, say why. If not, provide an example in which they are not similar and explain what happened.

(b) Find examples of two matrices with the same eigenvalues (counting multiplicities) that are similar.
(c) Find examples of two other matrices with the same eigenvalues (counting multiplicities) that are not similar.
(d) If A and C have the same **distinct** eigenvalues, then are A and C similar? Explain your answer.

(4) (5.3 #4) (**Cramer's Rule** for solving square linear systems)
Let $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$ be an invertible matrix. Cramer's rule provides an explicit formula for the solution of $A\mathbf{x} = \mathbf{b}$. Recall that $A\mathbf{x} = \mathbf{b}$ looks like:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{1}$$

(a) First we will use properties of determinants to show that $x_1 = \frac{\det(B_1)}{\det(A)}$, where $B_1 = \begin{bmatrix} \mathbf{b} & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$. We proceed in steps:

(i) Recall the following facts about determinants, letting B be a matrix obtained from A via a row operation.
- If B is obtained by swapping rows of A, then $\det(B) = -\det(A)$.
- If B is obtained by multiplying a row of A by a constant c, then $\det(B) = c\det(A)$
- If B is obtained by adding a multiple of one row of A to another, then $\det(B) = \det(A)$.

Argue that the same properties hold true for column operations.
(ii) Find a relationship between $\det(B_1)$ and $\det(A)$ by making an appropriate substitution for $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ in equation (1) and using part (i).
(iii) Conclude that $x_1 = \frac{\det(B_1)}{\det(A)}$

(b) Use part (a) to argue that $x_i = \frac{\det(B_i)}{\det(A)}$ for all $i = 2, 3, \ldots, n$, where $B_i = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \end{bmatrix}$.

(5) ***Change of bases and diagonalization**
Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(a) Show that $\mathbf{u}_1 = (2, -1)^\top$ and $\mathbf{u}_2 = (1, 2)^\top$ are eigenvectors of A that form a basis for $\mathbb{R}^2$. Call this basis $\mathcal{U}$.
(b) Compute the diagonalization of $A = U\Lambda U^{-1}$.
(c) On graph paper, draw the grid in $\mathbb{R}^2$ given by the basis $\mathcal{U}$ and locate the points on this grid with coordinates $(1, 1)$ and $(0, 5)$ with respect to the basis $\mathcal{U}$.
(d) Let $\mathbf{v}$ be the point in $\mathbb{R}^2$ whose coordinates are $(3, 1)^\top$ with respect to the standard basis of $\mathbb{R}^2$. Find the coordinates of $\mathbf{v}$ in the basis $\mathcal{U}$. Call this $\mathbf{y}$ and plot this point in your picture from (c).
(e) Compute $A\mathbf{v}$ and find its coordinates in the basis $\mathcal{U}$. Call this $\mathbf{z}$ and plot this point in your picture from (c).
(f) Check that $\mathbf{z} = \Lambda\mathbf{y}$.
(g) In the basis $\mathcal{U}$ what is the effect of the linear transformation A on the coordinates of a point? Explain in words.