Lesson 9

Read chapter 6

The four fundamental spaces, properties and bases for $V^\perp$

Subspaces associated with a matrix

$A \in R^{m \times n}$

$\underset{R^n}{NULL(A)}$, $\underset{R^m}{COL(A)}$, $\underset{R^n}{ROW(A)}$, $\underset{R^m}{NULL(A^T)}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$Null(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^2 \;\middle|\; \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Every vector in Null(A) is $\perp$ to every vector in row(A).

$$Null(A^T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \;\middle|\; \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Every vector in $Null(A^T)$ is $\perp$ to every vectors in col(A)

Th Given $A \in \mathbb{R}^{m \times n}$

$$\text{NULL}(A) = \text{ROW}(A)^{\perp}$$

Equivalently $\text{NULL}(A)^{\perp} = \text{ROW}(A)$

Proof: $\text{NULL}(A), \text{Row}(A) \subseteq \mathbb{R}^n$

First we will show $\text{NULL}(A) \perp \text{Row}(A)$

Suppose $v \in \text{Null}(A)$ so

$$A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} v = \begin{bmatrix} r_1^T v \\ r_2^T v \\ \\ r_m^T v \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

This tells us v is perpendiculer to $r_1, r_2 \cdots r_m$ and since $\text{Row}(A) = \text{span}(r_1, r_2, \ldots r_m)$ v is $\perp$ to any vector in $\text{Row}(A)$ so $\text{NULL}(A) \perp \text{Row}(A)$, so $\text{NULL}(A) \subseteq \text{Row}(A)^{\perp}$

Now suppose $w \in \text{Row}(A)^{\perp}$ then

$$\begin{bmatrix} r_1^T \\ r_2^T \\ r_m^T \end{bmatrix} w = \begin{bmatrix} r_1^T w \\ r_2^T w \\ r_m^T w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ so } w \in \text{Null}(A)$$

Therefore $\text{Row}(A)^{\perp} \subseteq \text{Null}(A)$

Since we have shown $\text{Null}(A) \subseteq \text{Row}(A)^{\perp}$ and $\text{Row}(A)^{\perp} \subseteq \text{Null}(A)$, we have

$$\text{Null}(A) = \text{Row}(A)^{\perp}$$

Th: given $A \in R^{m \times n}$

$$Null(A^T) = Col(A)^\perp$$

$$\text{or } Null(A^T)^\perp = Col(A)$$

proof: apply the previous th to $B = A^T$

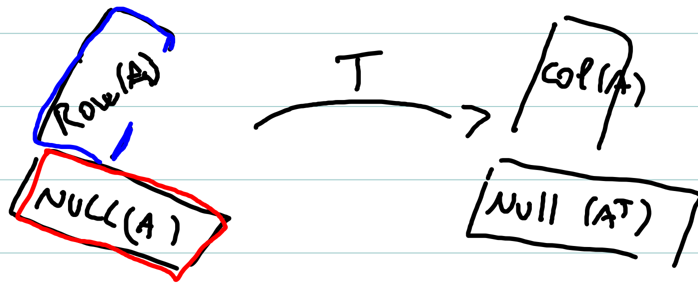
$Null(B) = row(B)^\perp$ so

$Null(A^T) = row(A^T)^\perp = Col(A)^\perp$

Given $A \in \mathbb{R}^{m \times n}$ look at

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T(v) = Av$$



Suppose $V \leq R^n$

Th

1) $V^{\perp} \leq R^n$

2) $V \cap V^{\perp} = \{0\}$

3) If S and T are subspaces of R^n with bases B_1 and B_2 and $S \cap T = \{\vec{0}\}$ then $B_1 \cup B_2$ is linearly independent

4) $\dim V^{\perp} = n - \dim V$

5) If $V \leq R^n$ any vector $w \in R^n$ can be written as $w = v + v_{\perp}$ with $v \in V$ and $v_{\perp} \in V^{\perp}$ in a unique way

6) $(V^{\perp})^{\perp} = V$

1) $V^{\perp} \leq \mathbb{R}^n$

Proof

Suppose $w_1 \in V^{\perp}$, $w_2 \in V^{\perp}$ we need to show $k w_1 \in V^{\perp}$, $w_1 + w_2 \in V^{\perp}$, $0 \in V^{\perp}$:

$v^T (k w_1) = k v^T w_1 = 0$ for all v in V

$v^T (w_1 + w_2) = v^T w_1 + v^T w_2 = 0$ for all v in V

$v^T \cdot 0 = 0$ for all v in V

2) $V \cap V^{\perp} = \{\vec{0}\}$

Proof:

suppose $v \in V \cap V^{\perp}$ then $v^T \cdot v = \|v\|^2 = 0$

so $v = \vec{0}$

3) If S and T are subspaces of R^n with bases B_1 and B_2 and $S \cap T = \{\vec{0}\}$ then $B_1 \cup B_2$ is linearly independent

Proof:

Suppose $B_1 = b_1 \dots b_s$ $\quad$ $B_2 = c_1 \dots c_t$

Consider $h_1 b_1 + \dots + h_s b_s + k_1 c_1 + \dots + k_t c_t = 0$ then

$$\underbrace{h_1 b_1 + \dots + h_s b_s}_{\in S} = \underbrace{-k_1 c_1 \dots - k_t c_t}_{\in T}$$

then $h_1 b_1 + \dots + h_s b_s = \vec{0}$ so $h_1 = h_2 \dots = h_s = 0$

$-k_1 c_1 - \dots - k_t c_t = \vec{0}$ so $-k_1 = \dots = -k_t = 0$

and $k_1 = k_2 \dots = k_t = 0$

4) $\dim V^{\perp} = n - \dim V$

Proof

Given V let $b_1 \dots b_k$ be a basis for V

then $V = \text{row}(A)$ where $A = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_k^T \end{bmatrix}$ A is $k \times n$

$V^{\perp} = \text{Null}(A)$ and by rank nullity th:

$$\underbrace{\dim V^{\perp}}_{\text{nullity}(A)} + \underbrace{\dim V}_{\text{rank}(A)} = n$$

5) If $V \leq \mathbb{R}^n$ any vector $w \in \mathbb{R}^n$ can be written as $w = v + v_{\perp}$ with $v \in V$ and $v_{\perp} \in V^{\perp}$ in a unique way

Proof:

If $B_1 = b_1 \dots b_s$ is a basis for V and $B_2 = c_1 \dots c_t$ is a basis for $V^{\perp}$ Then $B_1 \cup B_2$ is a basis for $\mathbb{R}^n$ since it is:

independent by 3 has n vectors by 4

then any w in $\mathbb{R}^n$ can be written as

$$w = \underbrace{k_1 b_1 + \dots + k_s b_s}_{\substack{\text{in } V \\ \text{proj}(w)}} + \underbrace{h_1 c_1 + \dots + h_t c_t}_{\text{in } V^{\perp}}$$

why unique? suppose $w = v_1 + z_1 = v_2 + z_2$ with $v_1, v_2 \in V$ z_1, z_2 in $V^{\perp}$, then $v_1 - v_2 = z_2 - z_1$ in $V \cap V^{\perp}$ so $v_1 = v_2$ and $z_2 = z_1$

6) $(V^{\perp})^{\perp} = V$

Proof:

First since $V \perp V^{\perp}$ we know that $V \subseteq (V^{\perp})^{\perp}$

Then we will compere dimensions:

if $\dim V = k$, then $\dim V^{\perp} = n-k$ and

$\dim (V^{\perp})^{\perp} = n-(n-k) = k$

so $V \subseteq (V^{\perp})^{\perp}$ and $\dim V = \dim (V^{\perp})^{\perp}$

therefore $V = (V^{\perp})^{\perp}$

Given $V \leq \mathbb{R}^n$ $V = \text{span}(v_1 \dots v_k)$

How to find a basis for $V^\perp$:

Look at $A = \begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix}$ then

$V = \text{row}(A)$, $V^\perp = \text{NULL}(A)$

Find a basis for $\text{NULL}(A)$

Def: Given $b \in R^n$ and $V \leq R^n$
we know that we can write
$b = b_V + e$ with $b_V \in V$, $e \in V^\perp$
in a unique way.
b_V is called the orthogonal projection of
b on V.

$P_V : R^n \to R^n$

$P_V(b) = b_V$

P_V is a linear transformation
$P_V(b) = Ab$ for some matrix A
Does A have special properties?
(recall hw 2 #2)

$P_V(P_V(b)) = P_V(b_V) = b_V$

$A^2 b = b_V = Ab$

for all b in R^n. so

1) $A^2 = A$
2) $V = \text{col}(A) = \text{Range}(P_V)$
3) $\text{Null}(A) = V^\perp$
4) $A = A^T$ why?

Given x, y in R^n

$x = v_1 + w_1 \qquad v_1 \in V \qquad w_1 \in V^\perp$

$y = v_2 + w_2 \qquad v_2 \in V \qquad w_2 \in V^\perp$

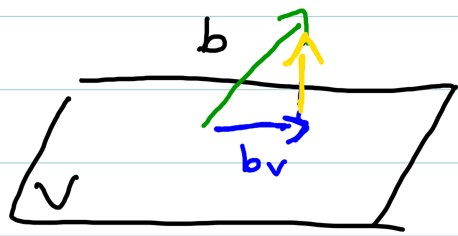
$x^T A y = (v_1^T + w_1^T)\ v_2 = v_1^T\ v_2$

$x^T A^T y = (Ax)^T y = v_1^T\ (v_2 + w_2) = v_1^T v_2$

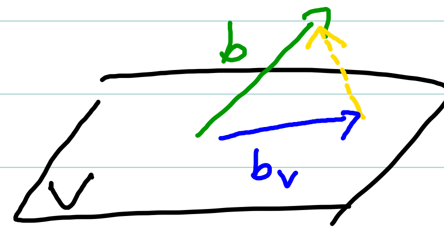
Therefore $A = A^T$

Projections

$A \quad \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad A^2 = A$



orthogonal



oblique

$NULL(A) = V^\perp$

$A = A^T$

$NULL(A) = ?$

Th : Suppose $A \in \mathbb{R}^{m \times n}$ and $\text{renk}(A) = n$
Then $A^T \cdot A$ is invertible.

A, A^T may not even be square so we cannot consider A^{-1}, $(A^T)^{-1}$ but $A^T A$ is invertible why?

Proof : $A^T A \in \mathbb{R}^{n \times n}$

we want to show $\text{NULL}(A^T A) = \{\vec{0}\}$
then $\text{renk}(A^T A) = n - \text{nullity}(A^T A) = n$
so $A^T A$ is invertible.

Suppose $A^T A y = 0$ then $y^T A^T A y = 0$
so $\|A y\|^2 = 0$ so $A y = \vec{0}$ so
$y \in \text{Null}(A)$. Since $\text{renk}(A) = n$
$\text{nullity}(A) = n - n = 0$ so $y = \vec{0}$