Lesson 8

Read chapter 4

dot product, norm, orthogonality
orthogonal spaces, orthogonal complement

Chapter 4

Def: If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in R^n$, $\|v\| = \sqrt{v_1^2+v_2^2+\cdots+v_n^2}$ length of v

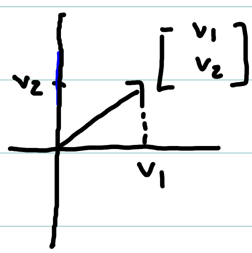
if $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in R^n$, $u \cdot v = u^T v = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} =$

$= u_1v_1 + u_2v_2 + \cdots + u_nv_n = v^T \cdot u$ (u "dot" v)

1. $v^T v = \|v\|^2$

2. $u^T v = \|u\| \cdot \|v\| \cos\theta$

Def: $u \perp v$ iff $u^T v = 0$

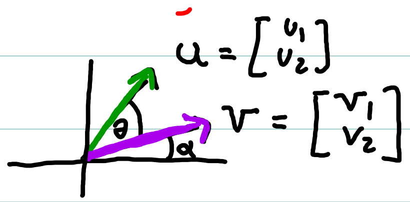


$\|v\| = \sqrt{v_1^2 + v_2^2}$ Pythagorean th

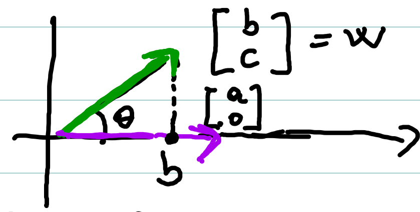
Not done in class. Read if you like, ask questions in office hours.

$$U \cdot V = U^T V = \|U\| \, \|V\| \cdot \cos\theta \quad : \quad \cos\theta = \frac{U_1V_1 + U_2V_2}{\|U\| \, \|V\|}$$

Justification: in R^2



rotate α

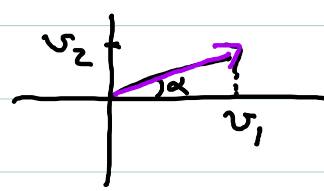


How do I get w from u? Clockwise rotation

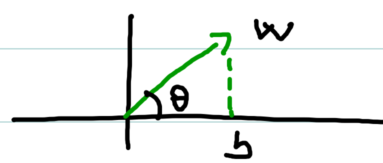
$$\begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}}_{\text{matrix of rotation}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \quad (*)$$

Also using trigonometry $\cos\alpha = \dfrac{v_1}{\|v\|}$

$$\sin\alpha = \frac{v_2}{\|v\|}$$



$$\cos\theta = \frac{b}{\|w\|} = \frac{b}{\|u\|}$$



From $(*)$ $\quad b = \cos\alpha \, u_1 + \sin\alpha \, u_2 = \dfrac{v_1}{\|v\|} u_1 + \dfrac{v_2}{\|v\|} u_2$

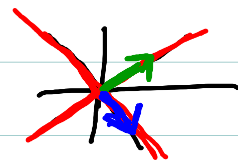
$$\cos\theta = \frac{v_1u_1 + v_2u_2}{\|v\| \, \|u\|} \qquad \text{so} \qquad u^T v = \|u\| \, \|v\| \cdot \cos\theta$$

Def: If V, W are subspaces of $\mathbb{R}^n$ then we say $V \perp W$ if given any $v \in V$, $w \in W$ we have $v \perp w$

Are the following orthogonal?

1) $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

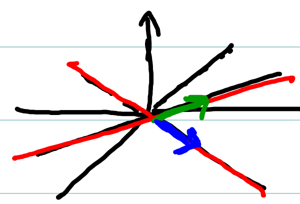
$[k \ \ k] \begin{bmatrix} h \\ -h \end{bmatrix} = kh - kh = 0$



yes

2) $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$ $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

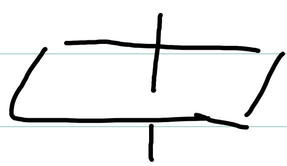
$[h \ \ h \ \ 0] \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = 0$



yes

3) $V = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$ $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

$(a, a, 0) \begin{pmatrix} b \\ -b \\ c \end{pmatrix} = ab - ab = 0$ yes



4) $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$ $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$[1 \ \ 1 \ \ 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \neq 0$ NO

Th Let $V = \text{span}(b_1, b_2, .. b_s) \leq \mathbb{R}^n$
and $W = \text{span}(c_1, c_2, \dots, c_t) \leq \mathbb{R}^n$
then $V \perp W \iff b_i \perp c_j$ for all
$1 \leq i \leq s$ and $1 \leq j \leq t$

proof $\Rightarrow$: suppose $V \perp W$ then any vector in V has to be $\perp$ to any vector in W so any b_i is $\perp$ to any c_j

$\Leftarrow$ Suppose $b_i \perp c_j$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$ and suppose $v \in V$ and $w \in W$ we want to show $v \perp w$:

$v = h_1 b_1 + h_2 b_2 \cdots + h_s b_s$; $w = k_1 c_1 + \cdots + k_t c_t$

$$v^T \cdot w = (h_1 b_1 + \cdots + h_s b_s)^T (k_1 c_1 + \cdots + k_t c_t)$$

$$= (h_1 b_1^T + \cdots + h_s b_s^T)(k_1 c_1 + \cdots + k_t c_t) =$$

$$= h_1 k_1 \, b_1^T c_1 + h_1 k_2 \, b_1^T c_2 + \cdots + h_1 k_t \, b_1^T c_t +$$

$$h_2 k_1 \, b_2^T c_1 + h_2 k_2 \, b_2^T c_2 + \cdots + h_2 k_t \, b_2^T c_T +$$

$$\cdots$$

$$h_s k_1 \, b_s^T c_1 + h_s k_2 \, b_s^T c_2 + \cdots + h_s k_t \, b_s^T c_T$$

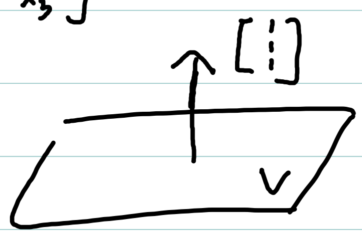
$$= 0$$

Therefore $v \perp w$

Are $V = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ and $W = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : x_1 + x_2 + x_3 = 0\right\}$ orthogonal?

if $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in W$ and $[h\ h\ h] \in V$

$[h\ h\ h] \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = h(x_1 + x_2 + x_3) = 0$ so yes



or

$W = \text{span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix} \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$ $V = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

$[1\ -1\ 0]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$ and $[0\ 1\ -1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$

Def : Given $V \leq \mathbb{R}^n$, the orthogonal complement of V is
$V^\perp = \{ w \in \mathbb{R}^n \mid w \perp x \text{ for all } x \text{ in } V \}$

① $V \perp V^\perp$

② $V^\perp$ the biggest possible subspace of $\mathbb{R}^n$ that is $\perp$ to V

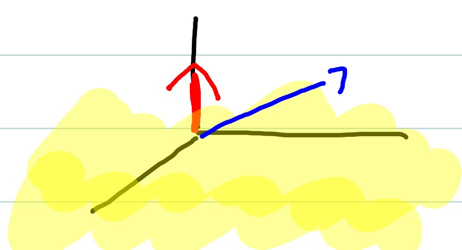
Example :

$V = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \quad W = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right);$

$W \perp V$ but $W \neq V^\perp$

because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin W$

Claim: $V^\perp = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$



Suppose $V \leq R^n$

Th: $V^\perp \leq R^n$

Th: $V \cap V^\perp = \{\vec{0}\}$

Th: If S and T are subspaces of R^n with bases B_1 and B_2 and $S \cap T = \{\vec{0}\}$ then $B_1 \cup B_2$ is linearly independent

Th: $\dim V^\perp = n - \dim V$

Th: If $V \leq R^n$ any vector $w \in R^n$ can be written as $w = v + v_\perp$ with $v \in V$ and $v_\perp \in V^\perp$ in a unique way

Th: $(V^\perp)^\perp = V$

Proofs next time

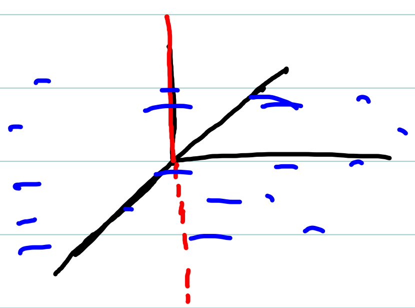
Ex in $\mathbb{R}^3$

$V = \text{span}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ $\quad V^\perp = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

How do I know ?

a) $\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \perp V$

b) $\dim \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 2 = 3-1 = \dim V^\perp$



$V^\perp$ xy plane =

$\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

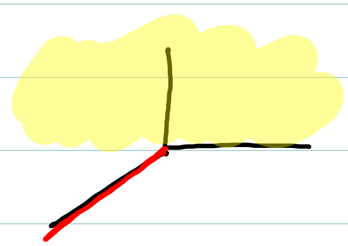
$$V = \text{span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) \qquad V^\perp = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

How do I know ?

a) $\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \perp V$

b) $\dim \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 1 = 3 - 2 = \dim V^\perp$

2



$V^\perp = x$ axis

$\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

Th: If $V, W \leq \mathbb{R}^n$ and $V \subseteq W$ and $\dim V = \dim W$ then $V = W$

Let $\dim V = \dim W = k$ let $B = b_1 \dots b_k$ be a basis for V then B is also a basis for W so $V = W$

Subspaces associated with a matrix

$A \in R^{m \times n}$

$\underset{R^n}{NULL(A)}$, $\underset{R^m}{COL(A)}$, $\underset{R^n}{ROW(A)}$, $\underset{R^m}{NULL(A^T)}$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in NULL(A) then

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: every vector

in NULL(A) is $\perp$ to every vector in ROW(A). This tells us $NULL(A) \perp ROW(A)$

Is $NULL(A) = ROW(A)^\perp$?

2) $NULL(A^T)$ $\qquad \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

So $NULL(A^T) \perp COL(A)$ is

$NULL(A^T) = COL(A)^\perp$?

Th Given $A \in R^{m \times n}$

$$NULL(A) = ROW(A)^\perp$$

Equivalently $NULL(A)^\perp = ROW(A)$

Proof: $NULL(A), Row(A) \subseteq R^n$

First we will show $NULL(A) \perp Row(A)$:

Suppose $v \in Null(A)$, then

$$A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} v = \begin{bmatrix} r_1^T v \\ r_2^T v \\ r_m^T v \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

This tells us v is perpendiculer to all the rows of A and therefore to any vector in Row(A) so

$NULL(A) \perp row(A)$, so $NULL(A) \subseteq Row(A)^\perp$

Now assume $w \in Row(A)^\perp$

then $A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} w = \begin{bmatrix} r_1^T w \\ r_2^T w \\ \vdots \\ r_m^T w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

so $w \in NULL(A)$ therefore $Row(A)^\perp = NULL(A)$

Since $NULL(A) \subseteq Row(A)^\perp$ and $Row(A)^\perp \subseteq NULL(A)$ then

$NULL(A) = ROW(A)^\perp$

Th Given $A \in R^{m \times n}$

$$NULL(A^T) = Col(A)^{\perp}$$

Equivalently $NULL(A^T)^{\perp} = Col(A)$

Proof:

Let $B = A^T$ and apply the previous th to B $\quad null(B) = row(B)^{\perp}$

$$null(A^T) = row(A^T)^{\perp}$$

Since $row(A^T) = col(A)$, we have

$$null(A^T) = col(A)^{\perp}$$