Lesson 5

Read chapter 2
Reed Chepter 3

Finish proof from last time
Fibonacci's sequence
General difference equation
Markov matrix

Th: if $A \in \mathbb{R}^{n \times n}$, $A = (a_{ij})$ then

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)}$$

π ranges over all n-permutations.

No proof

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Det A = is the sum of $4! = 24$ terms. $a_{11} \cdot a_{23} \cdot a_{32} \cdot a_{44}$ is one such term.

In this case π: 1 3 2 4

Th: Given $A \in \mathbb{R}^{n \times n}$, $\lambda_1, \lambda_2, \cdots \lambda_n$ the eigenvalues of A in $\mathbb{C}$ (not necessarily distinct), then

$p(\lambda) = k_n \lambda^n + k_{n-1} \lambda^{n-1} + \cdots + k_1 \lambda + k_0$ with

1) $k_0 = \det(A)$
2) $k_n = (-1)^n$
3) $k_{n-1} = (-1)^{n-1} \operatorname{tr}(A)$

Proof:

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots a_{2n} \\ \\ a_{n1} & \cdot & \cdots\cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

$p(\lambda) = \det(A - \lambda I)$

1) $k_0 = p(0) = \det(A - 0I) = \det(A)$

2) and 3):

By th above $p(\lambda) = \sum_{\pi} \operatorname{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots \pi_{n\pi(n)}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots a_{2n} \\ a_{n1} & \cdot & \cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

$$p(\lambda) = \sum_{\pi} \operatorname{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots \pi_{n\pi(n)}$$

if $\underbrace{\pi : 1\,2\,3 \cdots n}_{\text{identity permutation}}$

$$\begin{matrix} 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 2 \\ 3 & \longrightarrow & 3 \\ \vdots & & \vdots \\ n & \longrightarrow & n \end{matrix}$$

$\operatorname{sign}(\pi)\, b_{1\pi(1)} \cdots b_{n\pi(n)}$

$b_{11}\, b_{22} \cdots b_{nn}$ $= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) =$

Term of degree λ^{n-1}

$a_{11}\,(-\lambda)(-\lambda)\cdots(-\lambda) = (-1)^{n-1} a_{11}\lambda^{n-1}$

$(-\lambda)\,a_{22}\,(-\lambda)\cdots(-\lambda) = (-1)^{n-1} a_{22}\lambda^{n-1}$

$(-\lambda)(-\lambda)\,a_{33}\cdots(-\lambda) = (-1)^{n-1} a_{33}\lambda^{n-1}$

$\cdots$

$(-\lambda)(-\lambda)\cdots(-\lambda)\,a_{nn} = (-1)^{n-1} a_{nn}\lambda^{n-1}$

$= (-1)^n \lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1} +$ stuff of degree at most λ^{n-2}

What do the other terms $\text{sign}(\pi)\, b_{1\pi(1)} b_{2\pi(2)} \cdots \pi_{n\pi(n)}$ look like, if π is not the identy permutation?

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots a_{2n} \\ & & & \\ a_{n1} & \cdot & \cdots\cdots\cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

Consider $\text{sign}(\pi)\, b_{1\pi(1)} b_{2\pi(2)} \cdots b_{n\pi(n)}$:

if π is not the identity then there is i with $\pi(i) = j \neq i$. Then $\pi(j) = k \neq j$

Look at $\text{sign}(\pi)\, b_{1\pi(1)} b_{2\pi(2)} \cdots b_{i\pi(i)} \cdots b_{j\pi(j)} \cdots b_{n\pi(n)}$

b_{ij} b_{jk} are not diagonal elements

$$b_{ij} = a_{ij} \qquad b_{jk} = a_{jk}$$

no λ here

Therefore $\text{sign}(\pi)\, b_{1\pi(1)} b_{2\pi(2)} \cdots b_{n\pi(n)}$ is a polynomial in λ of degree $\leq n-2$

so $$p(\lambda) = \sum_{\pi} \text{sign}(\pi)\, b_{1\pi(1)} b_{2\pi(2)} \cdots b_{n\pi(n)}$$

$$= \underbrace{(-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1}}_{\text{from identy permutation}} + \text{stuff of degree at most } \lambda^{n-2}$$

Do not use this general fact in the hw. Do the calculations for a 3x3 matrix yourself.

Th: given $A \in \mathbb{R}^{n \times n}$ with complex eigenvalues $\lambda_1, \lambda_2 \dots \lambda_n$ (not necessarily distinct) then

$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$, $tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$

proof: by the previous th

$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} Tr(A) \lambda^{n-1} + \dots + \det(A)$,

then in $\mathbb{C}$

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= (-1)^n \lambda^n + (-1)^n (-1)(\lambda_1 + \lambda_2 + \dots + \lambda_n) \lambda^{n-1} + \dots + (-1)^n (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

For hw 2 Problem 1 a) i) and 1 b) ii) give a different argument for diagonalizable matrices

Ch 2

Def: Fibonacci's sequence

$F_0 = 0$

$F_1 = 1$

$F_{n+2} = F_{n+1} + F_n \qquad n+2 \geq 2$ Recursive definition

What is F_5? Linear, constant coefficients recursive difference equation

$F_2 = F_1 + F_0 = 1 + 0 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

Fibonacci's sequence using matrices

$F_0 = 0$

$F_1 = 1$

$F_{n+2} = F_{n+1} + F_n \qquad n+2 \geq 2$

We want:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What should a, b, c, d be?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} a F_{n+1} + b F_n \\ c F_{n+1} + d F_n \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$$

so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$F_0 = 0$

$F_1 = 1$ Fibonacci's sequence

$F_{n+2} = F_{n+1} + F_n$

Using matrices:

$$\underbrace{\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}}_{w_{n+1}} = \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{w_n} \quad : \quad \boxed{w_{n+1} = A\, w_n}$$

also

$$\underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{w_n} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}}_{w_{n-1}} \qquad w_n = A\, w_{n-1}$$

Check: $$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} \overbrace{F_n + F_{n-1}}^{\text{two consecutive Fibonacci's numbers}} \\ F_n \end{bmatrix} = \begin{bmatrix} \overbrace{F_{n+1}}^{\text{next Fibonacci}} \\ F_n \end{bmatrix}$$

$$w_n = A\, w_{n-1} = A\,A\, w_{n-2} = A\,A\,A\, w_{n-3} = \cdots = A^n w_0 \,, \quad w_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$w_n = A^n w_0$ Is A diagonalizable?

yes!

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$\lambda^2 - \lambda - 1 = 0$ for $\lambda_1 = \frac{1+\sqrt{5}}{2} \simeq 1.618$ $\quad \lambda_2 = \frac{1-\sqrt{5}}{2} \simeq -0.618$

Note : $\lambda_1 \lambda_2 = -1$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$E_{\lambda_1} = \text{Null}\begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix} = \text{Null}\begin{pmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{pmatrix}$$

reduce to echelon form

$$\begin{pmatrix} 1 & -\lambda_1 \\ \lambda_2 & 1 \end{pmatrix} \quad r_2 - \lambda_2 r_1 \to r_2 \quad \begin{pmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{pmatrix}$$

$$E_{\lambda_1} = \text{span}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}\right)$$

similar calculations show $E_{\lambda_2} = \text{span}\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$E_{\lambda_1} = \text{span}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}\right) \qquad E_{\lambda_2} = \text{span}\left(\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right)$$

$$A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} = PDP^{-1}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

So how do we compute F_n using this?

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$A^n = PDP^{-1} \cdot PDP^{-1} \cdot \ldots \cdot PDP^{-1} = PD^nP^{-1}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1^n & 0 \\ 0 & \alpha_2^n \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1^n c_1 \\ \alpha_2^n c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^{n+1} c_1 + \alpha_2^{n+1} c_2 \\ \alpha_1^n c_1 + \alpha_2^n c_2 \end{bmatrix}$$

$$F_n = \lambda_1^n c_1 + \lambda_2^n c_2$$

Note $|\lambda_2| < 1$ so $\lim_{n \to +\infty} \lambda_2^n \cdot c_2 = 0$

For big n $\quad F_n \approx \lambda_1^n c_1$

what are c_1 and c_2 ?

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

To find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ you can solve

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 = \frac{1}{\lambda_1 - \lambda_2} \qquad c_2 = -\frac{1}{\lambda_1 - \lambda_2}$$

Recall : $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\lambda_1 - \lambda_2 = \sqrt{5}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Golden ratio : $\frac{1+\sqrt{5}}{2} = \alpha_1$

see video in canvas

$$\lim_{n\to+\infty} \frac{F_{n+1}}{F_n} = \frac{\alpha_1^{n+1} c_1 + \alpha_2^{n+1} c_2}{\alpha_1^n c_1 + \alpha_2^n c_n} = \alpha_1$$

In general

$G_0 = x_0$

$G_1 = x_1$

$G_{n+2} = a G_{n+1} + b G_n$

$$\begin{bmatrix} G_{n+2} \\ G_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}}_{B} \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = B^n \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

$$p_B(\lambda) = (a-\lambda)(-\lambda) - b = \lambda^2 - a\lambda - b$$

Eigenvalues: roots of $\lambda^2 - a\lambda - b = 0$

as long as this equation has 2 distinct roots α_1 α_2 then $G_n = b_1 \alpha_1^n + b_2 \alpha_2^n$ for some constants b_1, b_2.

In notes more general setting (Prop 2.2.1) why ? $B = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}^{-1}$

$$\begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \alpha_1^n & 0 \\ 0 & \alpha_2^n \end{bmatrix} \underbrace{\begin{bmatrix} U_1 & U_2 \end{bmatrix}^{-1} \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}}_{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}$$

$$= c_1 \alpha_1^n U_1 + c_2 \alpha_2^n U_2$$

$$G_n = \underbrace{c_1 U_{12}}_{b_1} \alpha_1^n + \underbrace{c_2 U_{22}}_{b_2} \alpha_2^n$$

Last year's midterm

Problem 2. Consider the sequence defined by:

$$G_0 = 0,\ G_1 = 0,\ G_2 = 1,\ G_{k+2} = 6\,G_{k+1} - 11\,G_k + 6\,G_{k-1} \text{ for } k+2 \geq 3$$

1. Find a matrix A such that
$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \\ G_k \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \\ G_{k-1} \end{bmatrix}.$$

$$A = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \\ G_k \end{bmatrix} = A^k \begin{bmatrix} G_2 \\ G_1 \\ G_0 \end{bmatrix}$$

2. Find an explicit formula for G_k . You can use the fact that the matrix A has eigenvalues 1,2,3 with eigenspaces
$$E_1 = \text{ span } (u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}),\ E_2 = \text{span}(u_2 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}),\ E_3 = \text{span}(u_3 = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix})$$
and that $= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$. Show your work.

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \\ G_k \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1/2 \\ -2^k \\ \frac{3^k}{2} \end{bmatrix} = \frac{1}{2} u_1 - 2^k u_2 + \frac{3^k}{2} u_3$$

$$G_k = \frac{1}{2} - 2^k + \frac{3^k}{2}$$

Ch 3

In a metropolitan area some percentage x of the population lives in the city, some percentage y lives in the suburbs. The total population is constant but every year 90 % city population stays in city

10 % city population moves to suburbs.

98 % suburbs population stays in suburbs.

2 % suburbs populati moves to city.

How can I represent this info using matrices?

How can I use linear algebra to derive useful info?

Idea : summarize the information using a matrix

$$\begin{array}{c} C \\ S \end{array} \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Two possibilities (states)

$S_1 = C$: in city $\quad S_2 = S$: in suburbs

changing at discrete time intervals : $t_0, t_1, t_2 \cdots$

Probabilities P_{ij} : probability that if in state j some year will be in state i the following year

Something like this is called a Markov process

Ex: Brownian motion

$$\begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$$ is Markov matrix or transition matrix

Def: $A \in \mathbb{R}^{m\times n}$ is positive if all its entries are > 0

$A \in \mathbb{R}^{m\times n}$ is non negative " are ≥ 0

Def: $M \in \mathbb{R}^{n\times n}$ is Markov if it is non negative and all columns add to 1.

Ex $$M = \begin{bmatrix} p_{11} & p_{12} & & p_{1n} \\ p_{21} & p_{22} & & p_{2n} \\ p_{31} & p_{32} & & p_{3n} \\ & & & \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix}$$

p_{ij} : probability that if in state j at time k, then in state i at time $k+1$

If M is Markov

$$\underset{1 \times m}{[1, 1, \dots 1]} \underset{m \times n}{M} = \underset{1 \times n}{[1, 1, \dots 1]}$$

$$\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbb{1}^T M = \mathbb{1}^T$$

FACTS ABOUT MULTIPLICATION:

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n$$

$$[x_1 \cdots x_n] \begin{pmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_n^T \end{pmatrix} = x_1 c_1^T + x_2 c_2^T + \cdots + x_n c_n^T$$

For Markov matrix

$$[1 \cdots 1] \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} = [1 \cdots 1]$$

Note $A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $\boxed{\text{so } \lambda = 1 \text{ is eigenvalue for } A^T \text{ and for } A}$

We know from hw 1 that A and A^T have the same eigenvalues

Therefore A has eigenvalue 1

Th : if A is a Markov matrix then $\lambda = 1$ is an eigenvalue for A.

Def $V \in \mathbb{R}^n$ is a probability vector if all of its entries are ≥ 0 and add up to $\underline{1}$

Ex $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ is a probability vector

Back to our example. Suppose this year (year 0) 70% of the population of the metropolitan area lives in the city and 30% in the suburbs, i.e initial state is $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$. Consider

$$\begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.636 \\ 0.364 \end{bmatrix}$$

Still a probability vector, what does it represent?

$$0.9 \times 0.7 + 0.02 \times 0.3 = 0.636$$

$$0.1 \times 0.7 + 0.98 \times 0.3 = 0.364$$

$$\underbrace{0.9 \times 0.7}^{\text{city population that stayed}} + \underbrace{0.02 \times 0.3}^{\text{suburb population that moved to city}} = 0.636 \quad \% \text{ in city in year 1}$$

$$\underbrace{0.1 \times 0.7}_{\text{city population moved to suburbs}} + \underbrace{0.98 \times 0.3}_{\text{suburb pop that stayed}} = 0.364 \quad \% \text{ in suburbs in year 1}$$

so if $w_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ is initial state

and $M = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$ is the matrix of the system

then

$$M \underbrace{w_0}_{\text{year zero}} = \underbrace{w_1}_{\text{year 1}}$$

What percentage of the population will live in the city in year n ?

year zero

$$M w_0 = w_1 \quad \swarrow \text{year 1}$$

$$M w_1 = M M w_0 = w_2 \quad \swarrow \text{year 2}$$

$$\cdots$$

What percentage of the population will live in the city in year n?

$$M^n w_0 = w_n \quad \swarrow \text{year } n$$

How can I compute this?

So we want to compute M^n. Is M diagonalizable?

What are the eigenvalues of

$$M = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix} \quad ?$$

$\lambda = 1$ with eigenvector $u_1 = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$ positive positive probability vector

$\lambda_2 = 0.88$ with eigenvector $u_2 = \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix}$ negative positive

Note: $|\lambda_2| < 1$

$$M = P \begin{bmatrix} 1 & 0 \\ 0 & 0.88 \end{bmatrix} P^{-1} \qquad P = \begin{array}{cc} v_1 & v_2 \\ \left[\begin{matrix} 1/6 & -1/6 \\ 5/6 & 1/6 \end{matrix}\right. & \left.\right] \end{array}$$

$$w_n = M^n w_0$$

$$w_n = P \begin{bmatrix} 1^n & 0 \\ 0 & .88^n \end{bmatrix} \underbrace{P^{-1} w_0}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$w_n = P \begin{bmatrix} 1 & 0 \\ 0 & 0.88^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{P} \begin{bmatrix} c_1 & 0 \\ 0 & c_2\, 0.88^n \end{bmatrix} = c_1 . v_1 + \underbrace{c_2 \cdot 0.88^n v_2}_{\to 0 \text{ when } n \to +\infty}$$

eigenvector for $\lambda = 1$

In the long run the system goes to $c_1 v_1$

What are c_1 and c_2?

$$P^{-1}\begin{bmatrix}0.7\\0.3\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

so

$$P\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0.7\\0.3\end{bmatrix}$$

Solve this system

$$\left[\begin{array}{cc|c}1/6 & -1/6 & 0.7\\ 5/6 & 1/6 & 0.3\end{array}\right] \quad \left[\begin{array}{cc|c}1 & -1 & 4.2\\ 5 & 1 & 1.8\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & -1 & 4.2\\ 0 & 6 & -19.2\end{array}\right]$$

$$c_2 = \frac{-19.2}{6}$$

$$c_1 = 4.2 - \frac{19.2}{6} = \frac{25.2-19.2}{6} = 1$$

remember this 1

$$W_n = U_1 + -\frac{19.2}{6}\, 0.88^n\, U_2$$

$$= \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} - \frac{19.2 \times 0.88^n}{6} \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{6}\left[1 + \frac{19.2 \times 0.88^n}{6}\right] \\ \frac{1}{6}\left[5 - \frac{19.2 \times 0.88^n}{6}\right] \end{bmatrix}$$

In the long run $W_n \approx U_1 = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$

$\frac{1}{6}$ of the population will live in the city, $\frac{5}{6}$ in suburbus.