Lesson 4

Read chapter 1
Reed Chapter 2

Characteristic polynomial and permutations

Fibonacci's sequence

In hw 2 problem 2

You need to prove $col(A) = E_1$, for a certain matrix A.

A common technique to prove that two sets A and B are equal is to show $A \subseteq B$ and $B \subseteq A$

H2 problem 3

In 208 vector space $= R^n$

BUT

Vector space = set of $\underbrace{\text{objects}}_{\text{vectors}}$ that can be added and multiplied by scalars.

$R^{m \times n}$ is a vector space.

Subspace of $R^{m \times n}$ is a subset of $R^{m \times n}$ that:

1) Contains 0 matrix 2) Is closed under addition 3) Is closed under scalar multiplication.

hw 2 problem 4

If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$

and $B = (b_{ij}) \in \mathbb{R}^{n\times t}$

and $A\cdot B = (c_{ij}) \in \mathbb{R}^{m\times t}$

then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\begin{array}{c} \\ i \\ \\ \end{array}\begin{pmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{pmatrix} \begin{array}{c}\begin{array}{c} j \\ \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}\end{array}\end{array} = \begin{array}{c} \\ i \\ \\ \end{array}\begin{array}{c} j \\ \begin{pmatrix} & & \\ & \cdot & \\ & & \end{pmatrix}\end{array}$$

A B C

c_{ij} is the dot product of $\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$ and

$\begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{nj} \end{bmatrix}$, the i^{th} row of A and the j^{th} column of A

Suppose $A \in R^{4\times4}$ and

$p(\lambda) = (\lambda-1)(\lambda-2)(\lambda^2+1)$

A has (real) eigenvalues $\lambda = 1, 2$

is A diagonalizable?

yes for sure

no for sure V

it depends on A

There are only 2 eigenspaces E_1 and E_2 both of dimension 1, so it is not possible to find a basis for R^4 consisting of eigenvectors of A

Suppose $A \in R^{4\times 4}$ and

$p(\lambda) = (\lambda-1)^2(\lambda-2)^2$

A has eigenvalues $\lambda = 1, 1, 2, 2$

is A diagonalizable?

yes for sure

no for sure

it depends on A x

If E_1 and E_2 both have dimension 2, they will have bases b_1, b_2 and c_1 c_2 respectively and $b_1\ b_2\ c_1 c_2$ is a basis for R^4 consisting of eigenvectors of A. Ex $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

If one of E_1, E_2 (or both) has dimension 1 A is not diagonalizable.

Ex $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Def: An n-permutation is a bijection

$$\pi : \{1, 2, \dots n\} \to \{1, 2, \dots n\}$$

bijection = one to one + onto

Ex: $\pi \quad \{1, 2, 3\} \to \{1, 2, 3\}$

$\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 3$

Table representation π: $\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix}$

Forget first row π: $2 \ 1 \ 3$

permutation = rearrangement of $1, 2, \cdots n$

How many different 3-$^\circ$ tetions are there?

$\pi : \{1, 2, 3\} \to \{1, 2, 3\}$

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First element can be 1, 2, 3 (3 Choicles)

1 2 3	1 3 2
2 1 3	2 3 1
3 1 2	3 2 1

for the second element we hece 2 choices

for the third 1 choice

Total : $3 \cdot 2 \cdot \underline{1}$

of n elements ?

$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

Def: π inverts pair $i\, j$ if $i < j$ and $\pi(i) > \pi(j)$

π is even / odd if it inverts an even/odd number of pairs

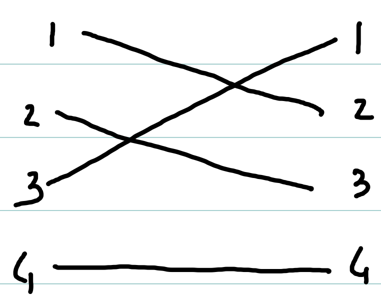
$$\text{sign}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd} \end{cases}$$

Example:

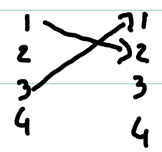
$\pi : \{1,2,3,4\} \to \{1,2,3,4\}$

$\pi : \quad 2 \; 3 \; 1 \; 4$

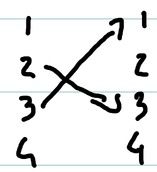
$\pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \quad \pi(4) = 4$



EVEN OR ODD ?



PAIR 1 3 inverted



PAIR 2 3 inverted

2 inverted pairs : EVEN

$\mathrm{sgn}(\pi) = 1$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Calculate det A expanding along the first column

$\det(A) =$

$a_{11}(a_{22}a_{33} - a_{23}a_{32})$

$- a_{21}(a_{12}a_{33} - a_{13}a_{32}) +$

$a_{31}(a_{12}a_{23} - a_{13}a_{22}) =$

$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} +$

$+ a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$

Note:

In every product we have one factor from row 1, one from row 2 and one from row 3. All factors are from different columns.

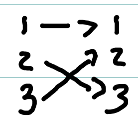
$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} +$

$+ a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} =$

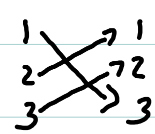
$= \sum_{\pi} \text{sign}(\pi)\, a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)}$ π ranges over all 6 3-permutations

Let's just check a couple. Look at

$-a_{11}\, a_{13}\, a_{12}$:

π : 1 3 2 is  odd

Look at $a_{13}\, a_{11}\, a_{12}$:

σ : 3 1 2  even

Th: if $A \in \mathbb{R}^{n \times n}$, $A = (a_{ij})$ then

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)}$$

π ranges over all n-permutations.

No proof

Hw 2 problem 1

1) a) ii) you can use the hint

$\det(A-\lambda I) = (-1)^n (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$

with no proof.

1 b) iii) Do explicit calculations without using the theorem in the next page

$p(\lambda) = \det(A-\lambda I)$

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = (b_{ij})$$

① Write $p(\lambda) = \det(A-\lambda I) =$

$= b_{11} b_{12} b_{13} + \cdots$ sum of 6 terms

② $p(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d$

show $a = (-1)$

write b in terms of the elements a_{ij} of matrix A

③ $p(\lambda) = -(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

expand to the form what is the coefficient of λ^2 if $p(\lambda)$ is written this way?

Th: Given $A \in R^{n\times n}$, $\lambda_1, \lambda_2, \cdots \lambda_n$ the eigenvalues of A in $\mathbb{C}$ (not necessarily distinct), then

$p(\lambda) = k_n \lambda^n + k_{n-1}\lambda^{n-1} + \cdots + k_1\lambda + k_0$ with

1) $k_0 = \det(A)$

2) $k_n = (-1)^n$

3) $k_{n-1} = (-1)^{n-1}\, tr(A)$

Proof:

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots & a_{2n} \\ a_{n1} & \cdot & \cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

$p(\lambda) = \det(A - \lambda I)$

1) $p(0) = k_0 = \det(A - 0I) = \det(A)$

2) and 3):

By th above $p(\lambda) = \sum_{\pi} sign(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots \pi_{n\pi(n)}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots a_{2n} \\ \\ a_{n1} & \cdot & \cdots\cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

$$p(\lambda) = \sum_{\pi} \operatorname{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots \pi_{n\pi(n)}$$

if $\underbrace{\pi : 1\ 2\ 3 \cdots n}_{\text{identity permutation}}$

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 2 \\ 3 & \longrightarrow & 3 \\ \vdots & & \vdots \\ n & \longrightarrow & n \end{array}$$

$\operatorname{sign}(\pi)\, b_{1\pi(1)} \cdots b_{n\pi(n)}$

$b_{11}\, b_{22} \cdots b_{nn} \qquad = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) =$

$= (-1)^n \lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1} +$ stuff of degree at most λ^{n-2}

What do the other terms $\text{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots \pi_{n\pi(n)}$ look like, if π is not the identy permutation?

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots a_{2n} \\ a_{n1} & \cdot & \cdots\cdots\cdots & a_{nn}-\lambda \end{bmatrix} = (b_{ij})$$

Consider $\text{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots b_{n\pi(n)}$:

at least two b's contain no λ: why?

suppose $\pi(i)=j$ $(i \neq j)$ then $b_{i\pi(i)} = b_{ij}$ $(i \neq j)$ and b_{ij}'s not a diagonal entry of $A-\lambda I$ so b_{ij} is just a number, no λ in it

Now look at $\pi(j)$. $\pi(j) \neq j$, because $\pi(i)=j$ and two different inputs i and j, cannot produce the same output. so $b_{j\pi(j)}$ is also not a diagonal entry of $A-\lambda I$, so it has no λ.

Therefore $\text{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots b_{n\pi(n)}$ is a polynomial in λ of degree $\leq n-2$

so $p(\lambda) = \sum_{\pi} \text{sign}(\pi)\, b_{1\pi(1)}\, b_{2\pi(2)} \cdots b_{n\pi(n)}$

$$= \underbrace{(-1)^n \lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1}}_{\text{from identy permutation}} + \text{stuff of degree at most } \lambda^{n-2}$$

Do not use this general fact in the hw. Do the calculations for a 3x3 matrix yourself.

Th: given $A \in \mathbb{R}^{n \times n}$ with complex eigenvalues $\lambda_1 \lambda_2 \dots \lambda_n$ (not necessarily distinct) then $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$ (in hw 2), $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$

proof: by the previous th

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{Tr}(A) \lambda^{n-1} + \dots + \det(A),$$

then in $\mathbb{C}$

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

the term of degree $n-1$ is

$(-1)^{n-1} (\lambda_1 + \lambda_2 + \dots + \lambda_n)$

Therefore $\operatorname{Tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$

$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$ hw problem

For hw 2 Problem 1 a) i) and 1 b) ii) give a different argument for diagonalizable matrices

Ch 2

Def: Fibonacci's sequence

$F_0 = 0$

$F_1 = 1$

$F_{n+2} = F_{n+1} + F_n \qquad n+2 \geq 2$ Recursive definition

what is F_5? Linear, constant coefficients recursive difference equation

$F_2 = F_1 + F_0 = 1 + 0 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

Fibonacci 's sequence using metrices

$F_0 = 0$

$F_1 = 1$

$F_{n+2} = F_{n+1} + F_n \qquad n+2 \geq 2$

We want:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What should a, b, c, d be ?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} a F_{n+1} + b F_n \\ c F_{n+1} + d F_n \end{bmatrix}$$

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$