Lesson 3

Read chapter 1

Diagonalization : finish proof from last time
Similar matrices
hw 1 help

Def $A \in R^{n\times n}$ is diagonalizable if we can write
$A = P D P^{-1}$ for some P, and diagonal D

Th: $A \in R^{n\times n}$ is diagonalizable iff there is a basis for R^n made of eigenvectors of A, that is if $\lambda_1 \cdots \lambda_k$ are the distinct eigenvalues of A and
E_{λ_1} has basis B_1, E_{λ_2} has basis B_2
.... E_{λ_k} has basis B_k
then $B = B_1 \cup B_2 \cdots \cup B_k$ is a basis for R^n

Ex $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable (in R)
why?

no eigenvalues

Ex $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable why?

Only eigenvalue $\lambda = 1$
$E_1 = \text{Null}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ which has dimension 1

$p(\lambda) = (\lambda - 1)^2$ 1 has algebraic multiplicity 2 and geometric multiplicity 1 proof of th: first suppose A is diagonalizable i.e $A = PDP^{-1}$ for some P and diagonal D. since P is invertible, the columns of P are a basis for R.

$$P = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \qquad B = b_1 \cdots b_n$$

then $P = U_B^E \qquad P^{-1} = U_E^B$

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

We want to show

$b_1, b_2, \dots b_n$ are eigenvectors of A for eigenvalues $d_1, d_2, \dots d_n$:

$$A b_1 = P D P^{-1} b_1 = P D U_E^B b_1 = P D [b_1]_B$$
$$= P D \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 P \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 b_1$$

so b_1 is an eigenvector for A with eigenvalue d_1

Similarly $Ab_2 = d_2 b_2$, $Ab_3 = d_3 b_3$ .....

Viceversa suppose $A \in \mathbb{R}^{n \times n}$ and $B = b_1 \cdots b_n$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A for eigenvalues $\lambda_1 \cdots \lambda_n$ (not necessarily distinct), that is $A b_i = \lambda_i b_i$.

. We want to show A is diagonalizable:

Take $P = \begin{bmatrix} b_1 \cdots b_n \end{bmatrix}$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, then P is invertible

$$P D P^{-1} b_i = P D U_{\mathcal{E}}^{B} b_i = P D [b_i]_B = P D e_i$$

$$= P \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = P \lambda_i e_i = \lambda_i P e_i = \lambda_i b_i$$

so $A b_i = P D P^{-1} b_i$ for $i = 1 \cdots n$

Is it enough to conclude $A = P D P^{-1}$?

yes: we will show $A e_i = P D P^{-1} e_i$ for $i = 1 \cdots n$

Therefore all columns of A and $P D P^{-1}$ are the same

$e_1 = k_1 b_1 + k_2 b_2 + \cdots + k_n b_n$, for some scalars $k_1, k_2, \cdots k_n$.

$A e_1 = A(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n) = k_1 (A b_1) + k_2 (A b_2) + \cdots + k_n (A b_n)$ = first column of A

$P D P^{-1} e_1 = P D P^{-1} (k_1 b_1 + k_2 b_2 + \cdots + k_n b_n) =$
$k_1 (P D P^{-1} b_1) + k_2 (P D P^{-1} b_2) + \cdots + k_n (P D P^{-1} b_n)$
$=$ first column of $P D P^{-1}$

Then first column of A = first column of $P D P^{-1}$. Similarly we can get that all columns are equal

Def: $A, B \in \mathbb{R}^{n \times n}$ are similar if $A = PBP^{-1}$ for some P

We will write $A \sim B$

Note if $A = PBP^{-1}$ then $P^{-1}AP = B$ so $B = QAQ^{-1}$, where $Q = P^{-1}$

Th: similar matrices have the same characteristic polynomial, therefore the same eigenvalues.

Proof: Assume $A = PBP^{-1}$ then

$$p_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I) =$$
$$\det(PBP^{-1} - P\lambda I P^{-1}) =$$
$$\det(P(B - \lambda I)P^{-1}) =$$
$$\det(P)\det(B - \lambda I)\det(P^{-1}) =$$
$$\det(B - \lambda I) = \quad \text{since } \det(P^{-1}) = \frac{1}{\det(P)}$$
$$= p_B(\lambda)$$

What about eigenvectors? In general not the same.

Th: Suppose: $A \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times n}$ and

$A = PMP^{-1}$, with $P = [b_1 \cdots b_n]$. $B = b_1 \cdots b_n$ is a basis for $\mathbb{R}^n$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ $\quad T(v) = Av$

then

$$[T(v)]_B = M\,[v]_B$$

M describes T from the point of view of B

Proof: $\quad P^{-1}AP = M$ therefore

$$M\,[v]_B = P^{-1}AP\,[v]_B = P^{-1}A\,U_B^E\,[v]_B =$$

$$P^{-1}Av = U_E^B(Av) = [Av]_B$$
$$= [T(v)]_B$$

Similar matrices represent the same linear transformation with respect to different bases.

Note: if $M = D$ is diagonal, $A = PDP^{-1}$

Consider $T \; \mathbb{R}^n \to \mathbb{R}^n \quad T(v) = Av$

$T(v)$?? Maybe hard to understand intuitively how T operates.

BUT if $P = \begin{bmatrix} b_1 \cdots b_n \end{bmatrix}$, $B = b_1 \ldots b_n$ basis of $\mathbb{R}^n$

then $[T(v)]_B = D[v]_B$ and if $[v]_B = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$

$$D[v]_B = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} d_1 k_1 \\ d_2 k_2 \\ d_n k_n \end{bmatrix}$$

is easy to understand.

See hw 1 problem 4.

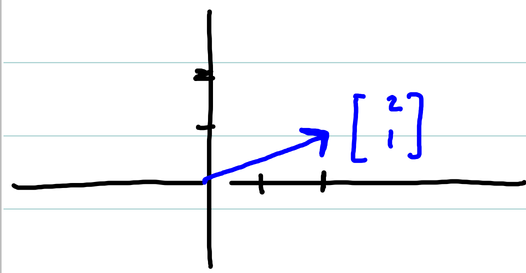
Example :

$$A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1}$$

Consider $T \quad R^2 \to R^2 \quad T(v) = Av$

what is $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$?



$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

but intuitively, without calculations,

I do not know what $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ is.

Let $B = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, basis for R^2.

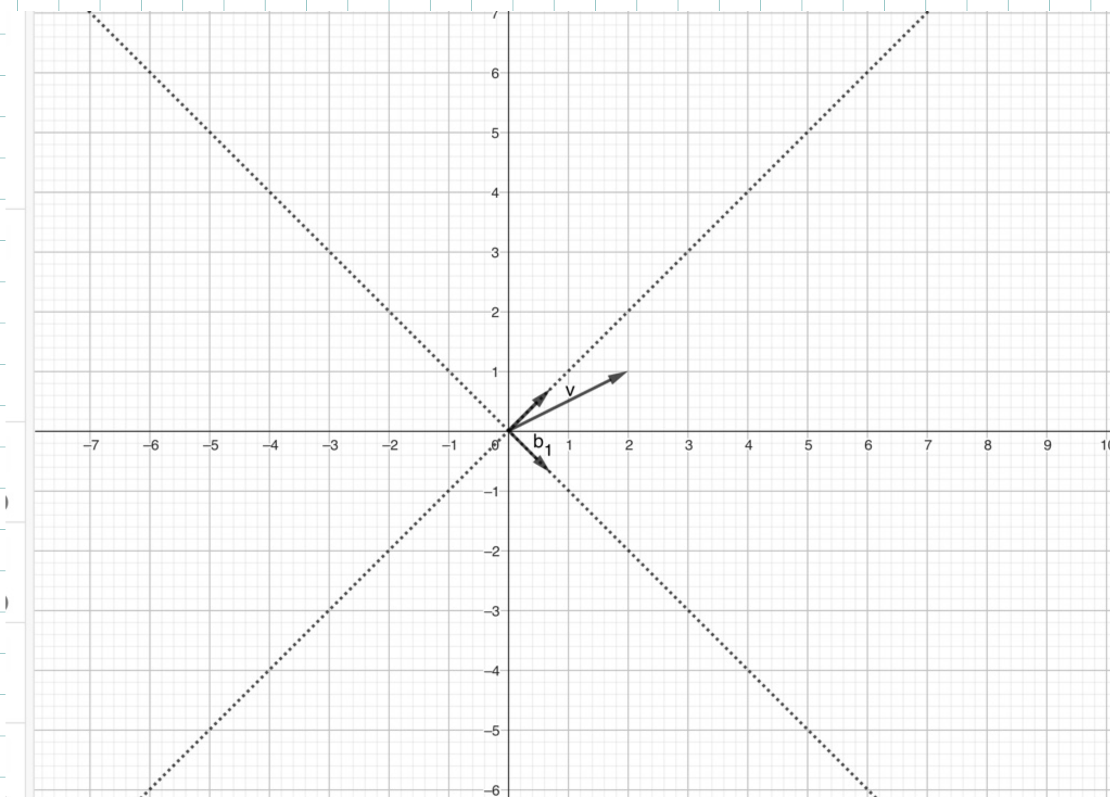
we know $\left[T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\right]_B = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_B$

how can I use this to have an intuitive

Understanding of T ?

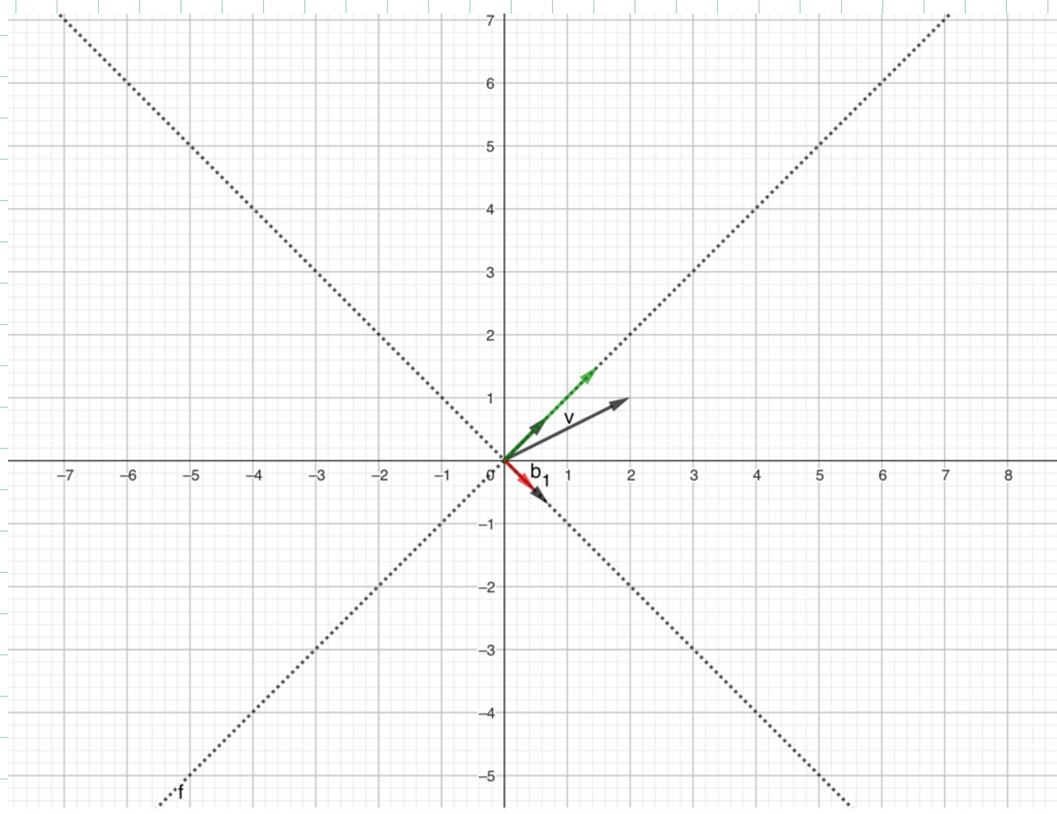
$$T\left(\left[\begin{bmatrix}2\\1\end{bmatrix}\right]_B\right) = \begin{bmatrix}-4 & 0\\0 & 2\end{bmatrix}\left[\begin{bmatrix}2\\1\end{bmatrix}\right]_B$$

$$B = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \quad \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$



Suppose $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = h \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$,

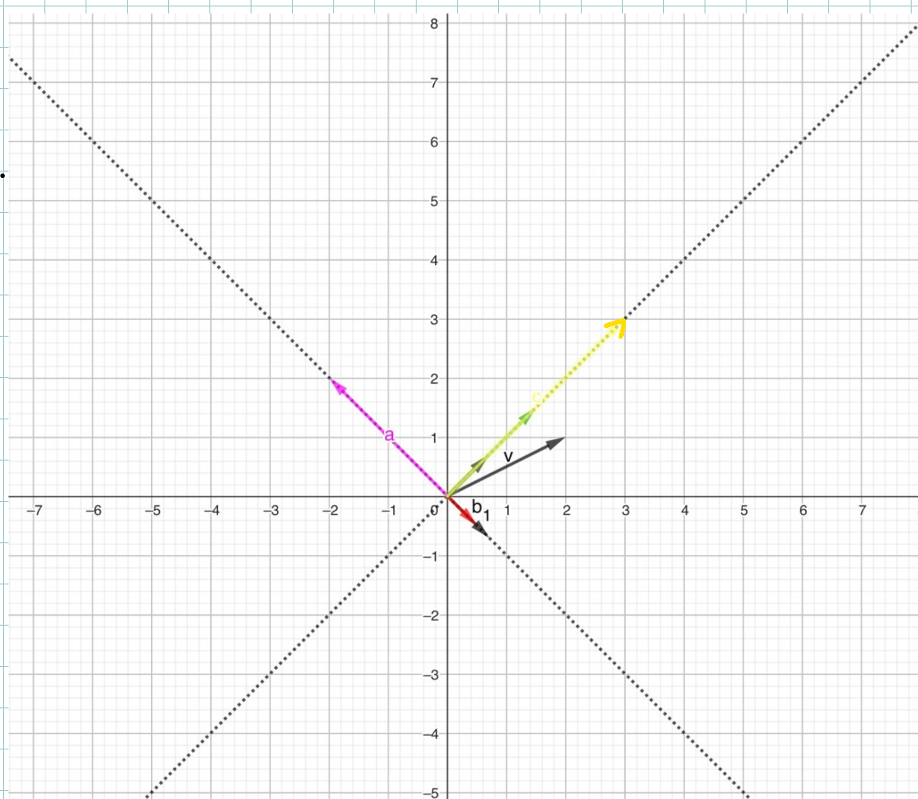
that is $\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]_B = \begin{bmatrix} h \\ k \end{bmatrix}$

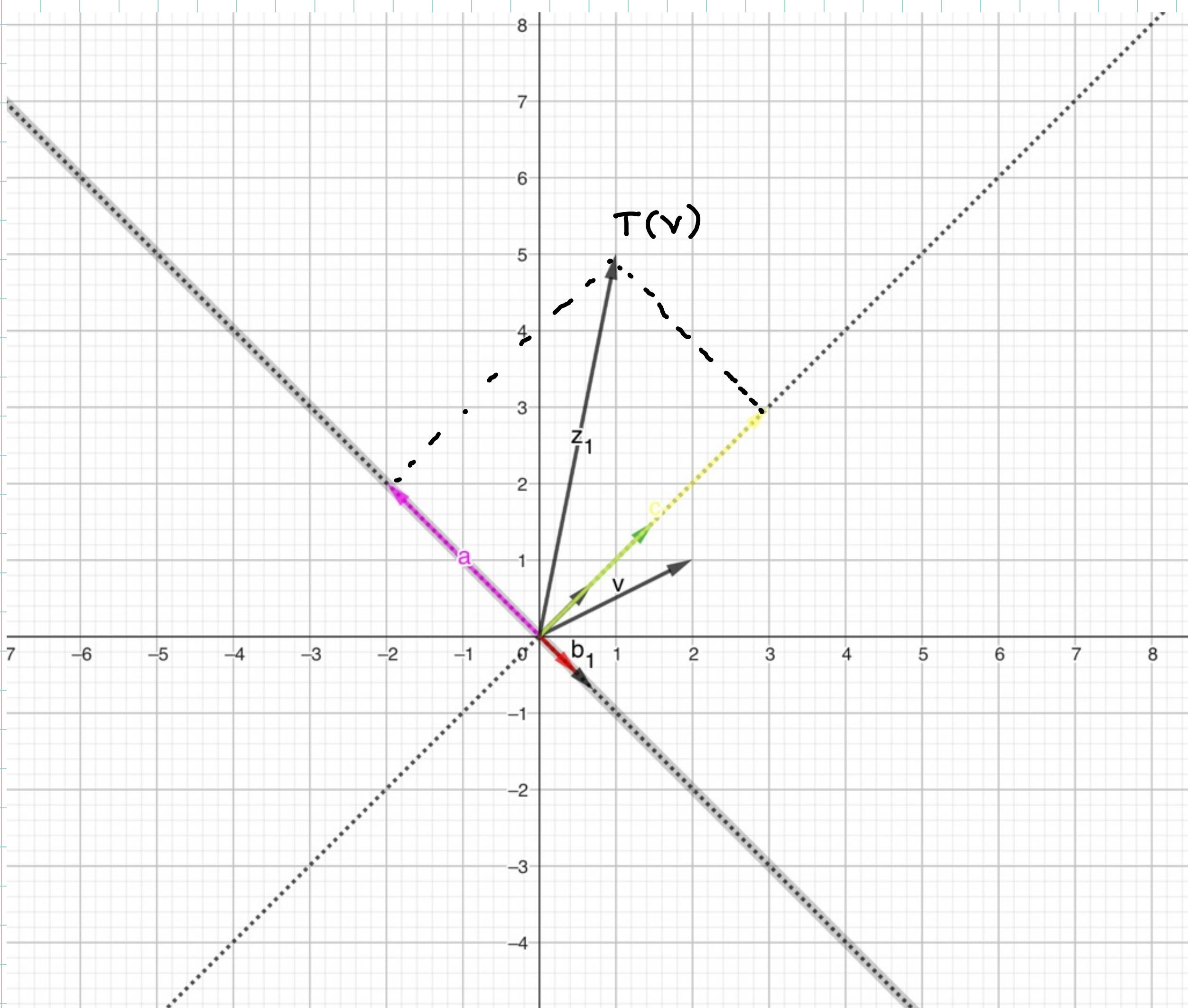


Then

$$\left[T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right]_{B} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -4h \\ 2k \end{bmatrix}$$

so $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = -4h \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + 2k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$




T(v)
z1
a
c
v
b1

you cannot use this table in hw without justification

	eigenvalue	eigenvector
If		
A	λ	v
Then		
A^2	λ^2	v
A^{-1}	$1/\lambda$	v
A^T	λ	?
A^k	λ^k	v
$A+3I$	$\lambda+3$	v
$p(A)$	$p(\lambda)$	v

Some proofs: Suppose $Av = \lambda v$ then

1) $AAv = A\lambda v = \lambda^2 v$

2) Suppose A invertible then

$A^{-1}Av = A^{-1}\lambda v$ so $v = \lambda A^{-1} v$ and

therefore $A^{-1} v = \frac{1}{\lambda} v$

(Recall that $\lambda \neq 0$ if A is invertible)

3) A and A^T have the same characteristic polynomial:

$$\det(A - \lambda I) = \det\left((A - \lambda I)^T\right) =$$
$$= \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I)$$

This gives us no information about eigenvectors. For most matrices, A and A^T have different eigenvectors.