# Lesson 21

Read Ch 8

rank 1 decomposition
Reduced SVD

rank 1 decomposition of A

A has rank r

$$\underset{m\times n}{A} = \underbrace{\begin{bmatrix} u_1 \cdots u_r & u_{r+1} \cdots u_m \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}}_{n\times n} \quad \text{by SVD}$$

then

rank 1 decomposition of A :

$$\underset{m\times n}{A} = \sigma_1 \underbrace{\underset{m\times 1}{u_1} \underset{1\times n}{v_1^T}} + \sigma_2 \underbrace{u_2 v_2^T} + \cdots + \sigma_r \underbrace{u_r v_r^T}$$

All these are $m\times n$ matrices of rank 1

(see hw 7)

Example:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} =$$

$$\sqrt{15} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Proof: let $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \to$ position $i$

$$\begin{bmatrix} u_1 \cdots u_r & u_{r+1} \cdots u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} e_i = U \overbrace{\begin{bmatrix} \sigma_1 v_1^T \\ \vdots \\ \sigma_r v_r^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{i\text{th column}} e_i =$$

$$= U \begin{bmatrix} \sigma_1 v_{1i} \\ \sigma_2 v_{2i} \\ \sigma_r v_{ri} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_1 v_{1i} u_1 + \sigma_2 v_{2i} u_2 + \cdots + \sigma_r v_{ri} u_r$$

$$(\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T) e_i =$$

$$= \sigma_1 v_{1i} u_1 + \sigma_2 v_{2i} u_2 + \cdots + \sigma_r v_{ri} u_r$$

so $i^{th}$ column of $U \Sigma V^T$ = $i^{th}$ column of $\sum_{k=1}^{r} \sigma_k u_k v_k^T$

This is true for $i = 1 \cdots n$

## Reduced SVD

$$\underset{m\times n}{A} = \underset{m\times m}{\begin{bmatrix} u_1 \cdots u_r & u_{r+1} \cdots u_m \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{bmatrix}} \quad \text{is SVD}$$

then

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$$

$$= \underset{m\times r}{\begin{bmatrix} u_1 \cdots u_r \end{bmatrix}} \underset{r\times r}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}} \underset{r\times n}{\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}} \quad \text{reduced SVD}$$

Example $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank $A = 3$

$A^T A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

$E_9 = \text{span}([0\ 0\ 0\ 1]^T)$

$E_4 = \text{span}([0\ 0\ 1\ 0]^T)$

$E_1 = \text{span}([0\ 1\ 0\ 0]^T)$

$E_0 = \text{span}([1\ 0\ 0\ 0]^T)$

$$A = \begin{bmatrix} \overbrace{\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}}^{\text{basis for col}(A)} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}\right\} \text{basis for row}(A) \\ \begin{matrix} 1 & 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

$u_i = \frac{1}{\sigma_i} A v_i$

$u_4$ in $\text{Null}(A^T)$ of length 1

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{SVD}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{reduced SVD}$$

Julia typically outputs the reduced SVD

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A = 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

rank 1 decomposition of A

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: in general the rank 1 decomposition (coming from the SVD) of a matrix A does NOT consist of matrices formed by keeping one column of A and replacing all others with 0

# Polar decomposition

Th: Every square matrix $A$ can be written as the product of an orthogonal matrix and a PSD matrix.

Proof

$$\underset{n\times n}{A} = U \Sigma V^T = \underbrace{U V^T}_{\text{orthogonal}} \underbrace{V \Sigma V^T}_{\text{PSD}}$$

$UV^T$ is orthogonal:

$$UV^T \cdot (UV^T)^T = UV^T V U^T = I$$

$V\Sigma V^T$ is PSD:

$V \Sigma V^T$ is symmetric and eigenvalues are the diagonal elements of $\Sigma$ so they are $\geq 0$

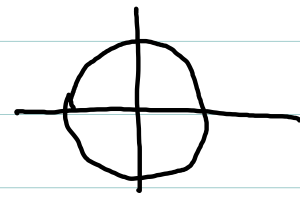
# Geometry of SVD

Def: Sphere of radius 1 in $R^n$

$$S^{n-1} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in R^n \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \right\}$$

Def: $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} + \cdots + \dfrac{x_n^2}{a_n^2} = 1$ is the equation of an hyperellipse in $R^n$

Def: $\dfrac{x_1^2}{a_1^2} + \cdots + \dfrac{x_r^2}{a_r^2} = 1 \quad x_{r+1} = \cdots = x_n = 0$ are the equations of an r dimensional hyperellipse in $R^n$

$S^1$  unit circle in $R^2$

has equation $x_1^2 + x_2^2 = 1$

$S^2$  unit sphere in $R^3$

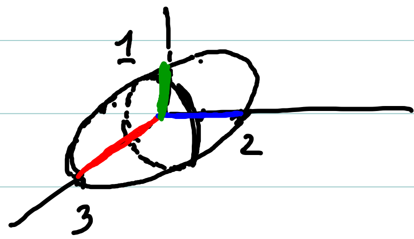
has equation $x_1^2 + x_2^2 + x_3^2 = 1$

$S^3$ unit sphere in $R^4$

has equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

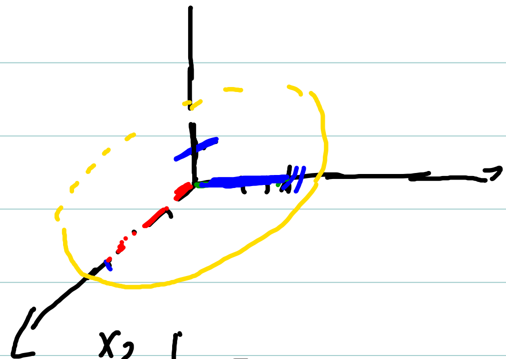
$$\frac{x_1^2}{9} + \frac{x_2^2}{4} + \frac{x_3^2}{1} = 1$$

is the equation of an hyperellipse in $\mathbb{R}^3$

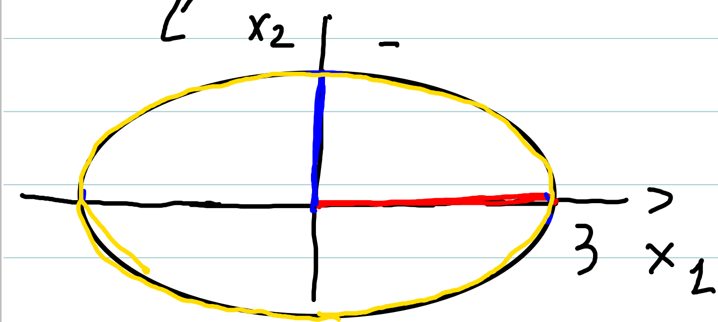



semi axes of this hyperellipse have lengths: 3, 2, 1

$$\frac{x_1^2}{9} + \frac{x_2^2}{4} = 1 \qquad x_3 = 0$$



Is an ellipse in the $x_1 x_2$ plane of $\mathbb{R}^3$




in $x_1 x_2$ plane semiaxes have lengths 3 and 2

Geometric interpretation : $T \; R^2 \to R^2 \quad T(w) = Aw$

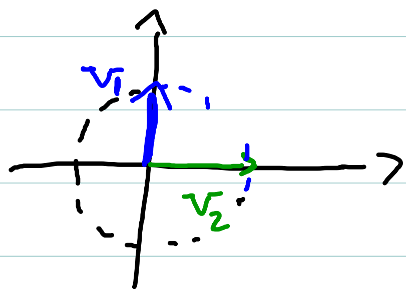
$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

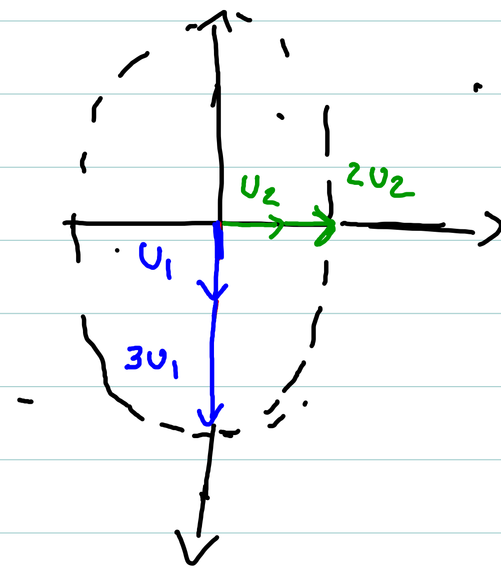

If $v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, $\alpha_1^2 + \alpha_2^2 = 1$

$$Av = \begin{bmatrix} 2\alpha_1 \\ -3\alpha_2 \end{bmatrix} \qquad \frac{(2\alpha_1)^2}{4} + \frac{(3\alpha_2)^2}{9} = 1$$

$v$ on circle $x^2+y^2=1$ $\quad Av$ on ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$




$u_1$ along major axis
$u_2$ along minor axis

$Av_1 = 3u_1 \qquad Av_2 = 2u_2$

Given $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

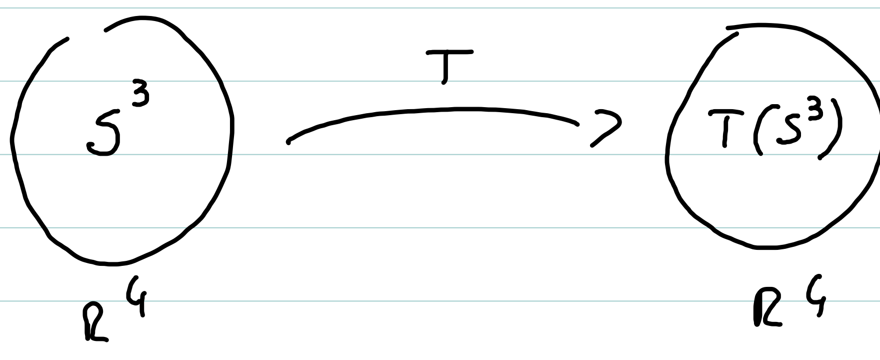
Consider $T \quad \mathbb{R}^4 \to \mathbb{R}^4 \quad T(v) = Av$

Unit sphere $S^3 = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x^2+y^2+z^2+t^2 = 1 \right\}$

SVD Tells what is $T(S^3)$

Fact it is an "hyperellipse"

Left singular vectors are along axes of this "hyperellipse"



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T \quad R^4 \to R^4$

$T(v) = Av$

$$v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \quad \text{on } S^3 \qquad \text{iff}$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1$$

$$T(v) = Av = \begin{bmatrix} \alpha_2 \\ 2\alpha_3 \\ 3\alpha_4 \\ 0 \end{bmatrix}$$

$$\alpha_2^2 + \frac{(2\alpha_3)^2}{4} + \frac{(3\alpha_4)^2}{9} = \alpha_2^2 + \alpha_3^2 + \alpha_4^2 \leq 1$$

If $v$ is on $S^3$ $T(v)$

is a vector of the form $\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$

with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ on or inside the

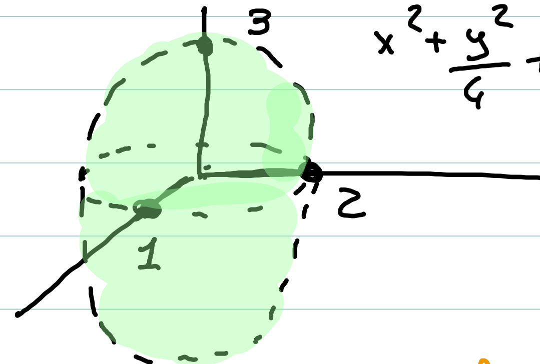
ellipsoid $x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{3^2} = 1$

$S^3 \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_2 \\ 2\alpha_3 \\ 3\alpha_4 \\ 0 \end{bmatrix}$ something in $R^4$ that projects onto

$\sigma_1 = 3$

$\sigma_2 = 2$

$\sigma_1 = 1$




$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad$ $Av_1 = 3u_1$ $\qquad u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ along major axis $z$

$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad$ $Av_2 = 2v_2$ $\qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ along $y$ axis

$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ $Av_3 = 3v_3$ $\qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ along $x$ axis

$v_4 \in \text{Span}(v_1 v_2 v_3)^\perp \qquad u_4 \in \text{Span}(u_1, u_2, u_3)^\perp$

$v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$T \quad R^4 \longrightarrow R^4$$

$$S^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1^2 + x_2^2 + x_3^2 + x_4^2 \right\}$$

is mapped by $T(v) = Av$ onto

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{array}{c} x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9} \leq 1 \\ x_4 = 0 \end{array} \right\}$$